

The Kupershmidt hydrodynamic chains and lattices

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Abstract

This paper is devoted to the very important class of hydrodynamic chains (see [9], [23], [24]) first derived by B. Kupershmidt in [14], later re-discovered by M. Blaszak in [4] (see also [21]). An infinite set of local Hamiltonian structures, hydrodynamic reductions parameterized by the hypergeometric function and reciprocal transformations for the Kupershmidt hydrodynamic chains are described.

In honour of Boris Kupershmidt

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1 Introduction

The Kupershmidt hydrodynamic chains (β and γ are arbitrary constants)

$$B_t^k = B_x^{k+1} + \frac{1}{\beta} B^0 B_x^k + (k + \gamma) B^k B_x^0, \quad k = 0, 1, 2, \dots \quad (1)$$

introduced in [14] recently were re-discovered in [4] for $\beta = 1/N$ and $\gamma = 1/M$, where N and M are integers (see also [20]). In this paper we consider three distinguished features of these hydrodynamic chains:

1. new explicit hydrodynamic reductions determined by the hypergeometric function;
2. infinitely many local and nonlocal Hamiltonian structures;
3. reciprocal transformations connecting the Kupershmidt hydrodynamic chains (1) with the distinct parameters β to each other.

Also we discuss in details the integrability of these hydrodynamic chains by the generalized hodograph method and related 2+1 quasilinear systems.

The paper is organized in the following order. In the second section the Gibbons equation describing a deformation of the Riemann mapping associated with the Kupershmidt hydrodynamic chain and deformations of the Riemann surfaces associated with corresponding hydrodynamic reductions is derived. In the third section we prove that the Gibbons–Tsarev system is common for all Kupershmidt hydrodynamic chains (irrespective of the distinct parameters β) as well as for the Benney hydrodynamic chain. In the fourth section infinitely many hydrodynamic reductions are found. In the fifth section infinitely many local Hamiltonian structures are constructed. Also nonlocal Hamiltonian structure connected with a metric of constant curvature is presented. In the sixth section the extended Kupershmidt hydrodynamic chains are constructed. In the seventh section linear transformations of independent variables preserving the Kupershmidt hydrodynamic chains are described. In the eighth section reciprocal transformations preserving the Kupershmidt hydrodynamic chains are found. In the ninth section the ideal gas dynamics as simplest two component hydrodynamic reduction of the Kupershmidt hydrodynamic chains is considered. In the tenth section the Kupershmidt hydrodynamic chains determined by special values of the parameter $\beta = N$ are derived. In the eleventh section multi-parametric solutions given by the generalized hodograph method are obtained for 2+1 quasilinear equations associated with the Kupershmidt hydrodynamic lattice.

2 The Gibbons equation and explicit hydrodynamic reductions

The Kupershmidt hydrodynamic chain (1) is a *linear* hydrodynamic chain with respect to discrete variable k (i.e. the r.h.s. of (1) is a *linear* function with respect to the discrete

variable k). Let us seek the moment decomposition (see [29]) in the form

$$B^k = \frac{1}{k + \gamma} \sum_{i=1}^N \varepsilon_i (b^i)^{\beta(k+\gamma)}, \quad \gamma \neq 0, -1, -2, \dots \quad (2)$$

Then (1) reduces to the hydrodynamic type system

$$b_t^i = \partial_x \left[\frac{(b^i)^{\beta+1}}{\beta+1} + \frac{B^0}{\beta} b^i \right], \quad i = 1, 2, \dots, N. \quad (3)$$

The generating function of conservation laws of this hydrodynamic reduction

$$p_t = \partial_x \left(\frac{p^{\beta+1}}{\beta+1} + \frac{B^0}{\beta} p \right) \quad (4)$$

can be obtained (see [28]) by the replacement $b^i \rightarrow p$. Simultaneously, (4) is the generating function of conservation laws for the Kupershmidt hydrodynamic chains (1).

Theorem 1: *The Gibbons equation*

$$\lambda_t - \left(p^\beta + \frac{B^0}{\beta} \right) \lambda_x = \frac{\partial \lambda}{\partial p} \left[p_t - \partial_x \left(\frac{p^{\beta+1}}{\beta+1} + \frac{B^0}{\beta} p \right) \right] \quad (5)$$

describes a deformation of the Riemann mapping $\lambda(\mathbf{b}; p)$ determined by the series

$$\lambda = q^{1-\gamma} + (1-\gamma) \sum_{k=0}^{\infty} \frac{B^k}{q^{k+\gamma}}, \quad \lambda \rightarrow \infty, \quad (6)$$

where $q = p^\beta$ and the coefficients $B^k(\mathbf{b}; p)$ satisfy (1).

Proof: If (4) and (3) are consistent, then

$$\partial_i q = q \frac{\partial_i B^0}{q - u^i} \left[\sum \frac{u^n \partial_n B^0}{q - u^n} - 1 \right]^{-1}, \quad (7)$$

where $u^i = (b^i)^\beta$ and $\partial_i = \partial / \partial u^i$. Since $\partial_i q = -\partial_i \lambda / \partial_q \lambda$ (this is a consequence of the consistency (3) with (5)), then the equation of the Riemann surface can be found in quadratures

$$d\lambda = q^{-\gamma} \left[\sum \frac{\varepsilon_n (u^n)^\gamma}{q - u^n} - 1 \right] dq - q^{1-\gamma} \sum \frac{\varepsilon_n (u^n)^{\gamma-1}}{q - u^n} du^n. \quad (8)$$

Thus, the equation of the Riemann surface

$$\lambda = q^{1-\gamma} + (1-\gamma) \sum \varepsilon_n \left(\frac{u^n}{q} \right)^\gamma F \left(1, \gamma, \gamma+1, \frac{u^n}{q} \right) \quad (9)$$

connected with (3) is parameterized by the hypergeometric function ${}_2F_1(a, b, c, z)$. Then, the substitution (2) in the above formula leads to the equation of the Riemann mapping (6) for (1). The theorem is proved.

If $\gamma = 0$, then the Kupershmidt hydrodynamic chain

$$B_t^k = B_x^{k+1} + \frac{1}{\beta} B^0 B_x^k + k B^k B_x^0, \quad k = 0, 1, 2, \dots \quad (10)$$

is connected with the equation of the Riemann mapping (6)

$$\lambda = q + \sum_{k=0}^{\infty} \frac{B^k}{q^k}, \quad \lambda \rightarrow \infty. \quad (11)$$

Definition 1: The Kupershmidt hydrodynamic chain (10) is said to be written in the *canonical* form.

Let us introduce the sub-index γ for the moments $B_{(\gamma)}^k$, which satisfy the Kupershmidt hydrodynamic chain (1)

$$\partial_t B_{(\gamma)}^k = \partial_x B_{(\gamma)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^k + (k + \gamma) B_{(\gamma)}^k B_x^0, \quad k = 0, 1, 2, \dots, \quad (12)$$

where $B_{(\gamma)}^0 \equiv B^0$. The invertible *polynomial* transformations $B_{(\gamma)}^k = B_{(\gamma)}^k(B^0, B^1, \dots, B^k)$ can be obtained by the comparison (6) with (11)

$$\lambda = q + \sum_{k=0}^{\infty} \frac{B^k}{q^k} = \left[q^{1-\gamma} + (1-\gamma) \sum_{k=0}^{\infty} \frac{B_{(\gamma)}^k}{q^{k+\gamma}} \right]^{\frac{1}{1-\gamma}}. \quad (13)$$

For instance,

$$B_{(\gamma)}^1 = B^1 - \frac{\gamma}{2} (B^0)^2, \quad B_{(\gamma)}^2 = B^2 - \gamma B^0 B^1 + \frac{\gamma(\gamma+1)}{6} (B^0)^3, \dots \quad (14)$$

If $\gamma = 1$, then the equation of the Riemann mapping (6) reduces to

$$\lambda = \ln q + \sum_{k=0}^{\infty} \frac{B_{(1)}^k}{q^{k+1}} \quad \Leftrightarrow \quad \lambda = q \exp \left(\sum_{k=0}^{\infty} \frac{B_{(1)}^k}{q^{k+1}} \right). \quad (15)$$

These above formulas are equivalent up to scaling $\lambda \rightarrow \exp \lambda$; since the Gibbons equation is a *linear* equation with respect to λ , any scaling $\lambda \rightarrow \tilde{\lambda}(\lambda)$ is admissible. Thus, the Kupershmidt hydrodynamic chains (10) and (12) are equivalent under the above invertible transformations (see [14]).

Remark: The rational hydrodynamic reductions of the Kupershmidt hydrodynamic chain (1) are found in [4] (two-component hydrodynamic reductions are described in [20]). The above hydrodynamic reductions (3) connected with the equation of the Riemann surface (9) are new, but still are not the most general. Most complicated (known at this moment) hydrodynamic reductions are considered below. Some of them are connected with the Hamiltonian structures of the Kupershmidt hydrodynamic chain (1).

If $\gamma = 0, -1, -2, \dots$, some of the moments $B_{(\gamma)}^k$ simplify. More precisely, the moment decomposition for these exceptional cases is given by (for $\gamma = -K$, $K = 0, 1, 2, \dots$)

$$B_{(-K)}^k |_{k \neq K} = \frac{1}{k-K} \sum_{i=1}^N \varepsilon_i (u^i)^{k-K}, \quad B_{(-K)}^K = \sum_{i=1}^N \varepsilon_i \ln u^i. \quad (16)$$

This degenerate case leads to the constraint $\Sigma \varepsilon_k = 0$. Let us emphasize again that this constraint $\Sigma \varepsilon_k = 0$ exists for the exceptional cases $\gamma = 0, -1, -2, \dots$ only (in all other cases it is still possible but an unessential restriction).

The Gibbons equation (5) (see [11], [28]) has three distinguished features.

1. If $\lambda = \text{const}$, then $\partial \lambda / \partial p \neq 0$ and (5) reduces to (4). Taking into account $q = p^\beta$ and substituting the inverse (the B rmann–Lagrange) series

$$q = \lambda - \sum_{k=0}^{\infty} \frac{Q^k(\mathbf{B})}{\lambda^k}, \quad q \rightarrow \infty$$

in (4) and (11), one can obtain an infinite series of polynomial conservation laws (see, for instance, the sections 5 and 7).

2. If $p = \text{const}$, then (5) reduces to the kinetic equation written in the Lax form (see [11], [36])

$$\lambda_t = \{\lambda, \hat{\mathbf{H}}\} \equiv \frac{\partial \hat{\mathbf{H}}}{\partial p} \frac{\partial \lambda}{\partial x} - \frac{\partial \hat{\mathbf{H}}}{\partial x} \frac{\partial \lambda}{\partial p},$$

where $\hat{\mathbf{H}} = p^{\beta+1}/(\beta+1) + B^0 p/\beta$ (cf. the flux of (4)). This Lax formulation can be derived as a dispersionless limit from R –matrix approach (see [4]) for $\beta = 1/N$

$$\lambda_t = \{\lambda, \hat{\mathbf{H}}\} \equiv q^{1-N} \left[\frac{\partial \hat{\mathbf{H}}}{\partial q} \frac{\partial \lambda}{\partial x} - \frac{\partial \hat{\mathbf{H}}}{\partial x} \frac{\partial \lambda}{\partial q} \right],$$

where $\hat{\mathbf{H}} = q^{N+1}/(N+1) + B^0 q^N$.

3. If $\partial \lambda / \partial p = 0$, then r.h.s. of (5) is vanished, and the hydrodynamic reductions (cf. (3))

$$b_t^i = \partial_x \left[\frac{(b^i)^{\beta+1}}{\beta+1} + \frac{B^0(\mathbf{b})}{\beta} b^i \right], \quad i = 1, 2, \dots, N, \quad (17)$$

can be written in the Riemann invariants $r^k = \lambda|_{\partial \lambda / \partial p = 0}$

$$r_t^i = \left[(p^i)^\beta + \frac{B^0(\mathbf{r})}{\beta} \right] r_x^i, \quad i = 1, 2, \dots, N, \quad (18)$$

where B^0 is a solution of some nonlinear PDE system describing N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable (see [12]), which we call the Gibbons–Tsarev system.

3 Universality of the Gibbons–Tsarev system

The consistency of the generating function of conservation laws (4) with the hydrodynamic type system (18) yields (cf. (7))

$$\partial_i q = q \frac{\partial_i B^0}{q^i - q},$$

where $q^i = (p^i)^\beta$ and $\partial_i \equiv \partial/\partial r^i$. The compatibility conditions $\partial_k(\partial_i q) = \partial_i(\partial_k q)$ yield the famous Gibbons–Tsarev system (see [12])

$$\partial_i \mu^k = \frac{\partial_i A^0}{\mu^i - \mu^k}, \quad \partial_{ik} A^0 = 2 \frac{\partial_i A^0 \partial_k A^0}{(\mu^i - \mu^k)^2}, \quad i \neq k, \quad (19)$$

where $\mu^i = q^i + B^0$ and the potential A^0 can be reconstructed in quadratures from $\partial_i A^0 = q^i \partial_i B^0$. Since the Gibbons–Tsarev system was derived for a description of hydrodynamic reductions of the Benney hydrodynamic chain (see [3])

$$A_t^k = A_x^{k+1} + k A^{k-1} A_x^0, \quad k = 0, 1, 2, \dots, \quad (20)$$

then hydrodynamic reductions of the different (*any* values of the parameter β) Kupershmidt hydrodynamic chains are the **same** as hydrodynamic reductions of the Benney hydrodynamic chain up to the aforementioned transformations $\partial_i A^0 = q^i \partial_i B^0$ and $\mu^i = q^i + B^0$.

4 Explicit hydrodynamic reductions

The Gibbons–Tsarev system (19) is integrable but a general solution is not found yet. However, in this section we shall be able to present the method allowing to find infinitely many explicit hydrodynamic reductions without solving the Gibbons–Tsarev system (19).

In the previous section we mentioned that any hydrodynamic reduction (18) of the Kupershmidt hydrodynamic chain (12) can be written in the form (17). However, this is not a unique choice. Following [12] an arbitrary hydrodynamic reduction (18) can be written via the first N moments $B_{(\gamma)}^k$

$$\begin{aligned} \partial_t B_{(\gamma)}^k &= \partial_x B_{(\gamma)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^k + k B_{(\gamma)}^k B_x^0, \quad k = 0, 1, 2, \dots, N-2, \\ \partial_t B_{(\gamma)}^{N-1} &= \partial_x B_{(\gamma)}^N(\mathbf{B}) + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^{N-1} + (N-1) B_{(\gamma)}^{N-1} B_x^0, \end{aligned}$$

where $B_{(\gamma)}^N(\mathbf{B})$ is a some function of the first N moments $B_{(\gamma)}^k$ compatible with the generating function of conservation laws (4). This hydrodynamic reduction significantly simplifies under the constraints $B^N = B^{N+1} = \dots = 0$ (see [4]). The similar constraints $B_{(\gamma)}^N = B_{(\gamma)}^{N+1} = \dots = 0$ are compatible with the Kupershmidt hydrodynamic chain (see [4] again) written in the form (12). The corresponding hydrodynamic type system

$$\begin{aligned} \partial_t B_{(\gamma)}^n &= \partial_x B_{(\gamma)}^{n+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^n + (n + \gamma) B_{(\gamma)}^n B_x^0, \quad n = 0, 1, 2, \dots, N-2, \\ \partial_t B_{(\gamma)}^{N-1} &= \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^{N-1} + (N-1 + \gamma) B_{(\gamma)}^{N-1} B_x^0. \end{aligned} \quad (21)$$

is connected with the equation of the Riemann surface (cf. (6))

$$\lambda = q^{1-\gamma} + (1-\gamma) \sum_{k=0}^{N-1} \frac{B_{(\gamma)}^k}{q^{k+\gamma}}. \quad (22)$$

Remark: The Kupershmidt hydrodynamic chains are equivalent under the invertible transformations $B_{(\gamma)}^k = B_{(\gamma)}^k(B^0, B^1, \dots, B^K)$, but the corresponding reductions (21) are different for an arbitrary values of parameter γ . It is obvious if to take into account (22) (cf. (13))

$$q + \sum_{k=0}^{N-1} \frac{B^k}{q^k} \neq \left[q^{1-\gamma} + (1-\gamma) \sum_{k=0}^{N-1} \frac{B_{(\gamma)}^k}{q^{k+\gamma}} \right]^{\frac{1}{1-\gamma}}.$$

An arbitrary ($K+M$ component) hydrodynamic reduction (18) can be written in the *mixed* form

$$\begin{aligned} \partial_t B_{(\gamma)}^k &= \partial_x B_{(\gamma)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^k + (k+\gamma) B_{(\gamma)}^k B_x^0, \quad k = 0, 1, 2, \dots, K-2, \\ \partial_t B_{(\gamma)}^{K-1} &= \partial_x B_{(\gamma)}^K(\mathbf{B}, \mathbf{b}) + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^{K-1} + (K-1+\gamma) B_{(\gamma)}^{K-1} B_x^0, \\ b_t^m &= \partial_x \left(\frac{(b^m)^{\beta+1}}{\beta+1} + \frac{B^0}{\beta} b^m \right), \quad m = 1, 2, \dots, M, \end{aligned}$$

where $B^K(\mathbf{B}, \mathbf{b})$ is a some function (of the punctures b^m and the first moments B^k), which must be compatible with the generating function of conservation laws (4). We avoid a derivation of a nonlinear PDE system on function $B_{(\gamma)}^K(\mathbf{B}, \mathbf{b})$, because the corresponding system is equivalent the Gibbons–Tsarev system (see the previous section). However, one particular solution can be found in the explicit form without this complicated analysis.

Main result of this section: a most general *explicit* hydrodynamic reduction found at this moment

$$\begin{aligned} \partial_t B_{(\gamma)}^k &= \partial_x B_{(\gamma)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^k + (k+\gamma) B_{(\gamma)}^k B_x^0, \quad k = 0, 1, \dots, K-2, \\ \partial_t B_{(\gamma)}^{K-1} &= \partial_x B_{(\gamma)}^K(\mathbf{b}) + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^{K-1} + (K-1+\gamma) B_{(\gamma)}^{K-1} B_x^0, \\ b_t^k &= \partial_x \left(\frac{(b^k)^{\beta+1}}{\beta+1} + \frac{B^0}{\beta} b^k \right), \quad k = 1, 2, \dots, M, \end{aligned} \tag{23}$$

is connected with the equation of the Riemann surface (cf. (9) and (22))

$$\lambda = \frac{q^{1-\gamma}}{1-\gamma} + \sum_{k=0}^{K-1} \frac{B_{(\gamma)}^k}{q^{k+\gamma}} + \sum_{m=1}^M \varepsilon_m \left(\frac{u^m}{q} \right)^{K+\gamma} F \left(1, K+\gamma, K+1+\gamma, \frac{u^m}{q} \right), \tag{24}$$

while all higher moments ($\gamma \neq -K, -K-1, -K-2, \dots$; see (16)) are given by (see (2))

$$B_{(\gamma)}^{K+n} = \frac{1}{K+n+\gamma} \sum_{m=1}^M \varepsilon_m (u^m)^{K+n+\gamma}, \quad n = 0, 1, 2, \dots \tag{25}$$

The equation of the Riemann surface (24) can be simplified for $\gamma = L_1/L_2$ (where L_1 and L_2 and integers), because a hypergeometric function degenerates in elementary functions. If $\gamma = -K, -K-1, -K-2, \dots$, then the extra constraint $\sum \varepsilon_m = 0$ appears.

If $\gamma = 1$, then the above hydrodynamic reductions (23) become *homogeneous* hydrodynamic type systems; if, for instance, $\gamma = 1$ and $K = 0$, then so-called the Schwarz-Christoffel type reduction (cf. [12]) is determined by

$$\lambda = q \prod_{k=1}^N \left(1 - \frac{u^k}{q}\right)^{-\varepsilon_k}.$$

Indeed, the hydrodynamic type system (23) is a hydrodynamic reduction of the Kupershmidt hydrodynamic chain (12). It is easy to verify taking into account that (23) is compatible with more deep reduction (2). It means that the hydrodynamic reduction (23) can be obtained from (12) by the moment decomposition (25) applied to the higher ($n \geq K$) moments $B_{(\gamma)}^n$ only. Then the first ($n < K$) moments $B_{(\gamma)}^n$ are natural field variables as well as the punctures b^k .

All other reductions can be obtained by different parametric and functional degenerations. The substitution of the Taylor series (we remember that $q = p^\beta$)

$$q = \sum_{k=0}^{\infty} q^k \lambda^k, \quad \lambda \rightarrow 0$$

in (4) yields

$$q_t^k = \frac{B^0}{\beta} q_x^k + q^k B_x^0 + \sum_{m=0}^k q^{k-m} q_x^m, \quad k = 0, 1, 2, \dots \quad (26)$$

Let us consider M such series *truncated* up to some numbers M_k

$$\partial_t q_{(k)}^m = \frac{B^0}{\beta} \partial_x q_{(k)}^m + q_{(k)}^m B_x^0 + \sum_{n=0}^m q_{(k)}^{m-n} \partial_x q_{(k)}^n, \quad m = 0, 1, \dots, M_k, \quad k = 0, 1, \dots, M.$$

If $M_k = 0$, this is the general case (23). Exceptional cases ($M_k \neq 0$) can be obtained by the *merging* of neighbouring singular points $q_{(k)}^0 \rightarrow q_{(k+1)}^0$. For instance, if $M_k = 1$ and $K = 0$, the Kupershmidt hydrodynamic chain (12) has a one parametric family of the Zakharov hydrodynamic reductions (the parameter γ is arbitrary for the each fixed parameter β)

$$b_t^i = \partial_x \left[\frac{(b^i)^{\beta+1}}{\beta+1} + \frac{B^0}{\beta} b^i \right], \quad c_t^i = \partial_x \left[\left((b^i)^\beta + \frac{B^0}{\beta} \right) c^i \right], \quad i = 1, 2, \dots, N$$

connected with the equation of the Riemann surface

$$\lambda = q^{1-\gamma} \left[1 + (1-\gamma) \sum_{k=1}^N \frac{(b^k)^{\beta\gamma-1} c^k}{q - (b^k)^\beta} \right], \quad \gamma \neq 1,$$

where

$$B_{(\gamma)}^n = \sum_{k=1}^N (b^k)^{\beta(n+\gamma)-1} c^k, \quad n = 0, 1, 2, \dots$$

If $\gamma = 1$, then

$$\lambda = \ln q + \sum_{k=1}^N \frac{(b^k)^{\beta-1} c^k}{q - (b^k)^\beta}.$$

Let us take for simplicity just one series (26) (i.e. $M = 1$) truncated by the number L (this is exactly the case considered in [4]). Then hydrodynamic reduction (23) reduces to

$$\begin{aligned} \partial_t B_{(\gamma)}^k &= \partial_x B_{(\gamma)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^k + (k + \gamma) B_{(\gamma)}^k B_x^0, \quad k = 0, 1, \dots, K-2, \\ \partial_t B_{(\gamma)}^{K-1} &= \partial_x B_{(\gamma)}^K(\mathbf{q}) + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^{K-1} + (K-1 + \gamma) B_{(\gamma)}^{K-1} B_x^0, \\ q_t^k &= \frac{B^0}{\beta} q_x^k + q^k B_x^0 + \sum_{m=0}^k q^{k-m} q_x^m, \quad k = 0, 1, \dots, L-1, \end{aligned}$$

where the function $B_{(\gamma)}^K(\mathbf{q})$ can be found from the linear PDE system

$$q \frac{\partial f}{\partial q} + \sum_{k=0}^{L-1} q^k \frac{\partial f}{\partial q^k} = 0, \quad q \frac{\partial f}{\partial q^k} - \sum_{m=0}^{L-k-1} q^m \frac{\partial f}{\partial q^{m+k}} = q^{1-K-\gamma} \frac{\partial B_{(\gamma)}^K(\mathbf{q})}{\partial q^k}, \quad k = 0, 1, \dots, L-1.$$

In this case the equation of the Riemann surface is given by

$$\lambda = \frac{q^{1-\gamma}}{1-\gamma} + \sum_{k=0}^{K-1} \frac{B_{(\gamma)}^k}{q^{k+\gamma}} + f(q^0, q^1, \dots, q^{L-1}; q).$$

5 The Hamiltonian structures

The theory of Hamiltonian operators for hydrodynamic chains starts from [15], where the first such Hamiltonian structure was derived, and from [6], where the Jacobi identity for local Poisson brackets was presented. If such a local Poisson bracket is written in the Liouville coordinates B^k

$$\{B^k, B^n\} = [\mathcal{W}^{kn}(\mathbf{B}) \partial_x + \partial_x \mathcal{W}^{nk}(\mathbf{B})] \delta(x - x'),$$

then the Jacobi identity reduced to the most compact form (see [27])

$$\begin{aligned} (\mathcal{W}^{ik} + \mathcal{W}^{ki}) \partial_k \mathcal{W}^{nj} &= (\mathcal{W}^{jk} + \mathcal{W}^{kj}) \partial_k \mathcal{W}^{ni}, \\ \partial_n \mathcal{W}^{ij} \partial_m \mathcal{W}^{kn} &= \partial_n \mathcal{W}^{kj} \partial_m \mathcal{W}^{in}. \end{aligned} \tag{27}$$

The very important class of the Poisson brackets determined by $\mathcal{W}^{kn} = \mathcal{W}_{(L)}^{kn}(B^0, B^1, \dots, B^{k+n-L})$, which we call the L -brackets, can be completely described (see [27]).

In this paper we deal with L -brackets.

Main statement of this section: The Kupershmidt hydrodynamic chains (1) has **infinitely** many *local* Hamiltonian structures enumerated by the index L for $L \geq 0$ and infinitely many nonlocal Hamiltonian structures if $L < 0$.

If $L = 0$. The important example of such a Poisson bracket is the Kupershmidt Poisson brackets (see [14])

$$\{B^k, B^n\} = [(\beta k + \gamma)B^{k+n}\partial_x + (\beta n + \gamma)\partial_x B^{k+n}]\delta(x - x') \quad (28)$$

connected with the Kupershmidt hydrodynamic chains (see below).

These Poisson brackets have the momentum B^0 only.

If $L = 1$. The famous example of such a Poisson bracket is the Kupershmidt–Manin bracket found in [15] for the Benney hydrodynamic chain (20)

$$\{B^k, B^n\} = [kB^{k+n-1}\partial_x + n\partial_x B^{k+n-1}]\delta(x - x').$$

This Poisson bracket has the momentum B^1 and the Casimir (annihilator) B^0 .

If $L \geq 1$, then corresponding Poisson brackets have L Casimirs. Since the Hamiltonian is $\bar{\mathbf{H}}_{L+1} = \int \mathbf{H}_{L+1}(B^0, B^1, B^2, \dots, B^{L+1})dx$, then the momentum can be chosen as $\bar{\mathbf{H}}_L = \int B^L dx$. The Casimirs can be chosen as $\bar{\mathbf{H}}_k = \int B^k dx$ ($k = 0, 1, 2, \dots, L-1$). Then the *auxiliary* (natural) restrictions (“normalization”) are

$$\begin{aligned} B^k &= \mathcal{W}_{(L)}^{Lk}(B^0, B^1, \dots, B^k), \quad k = 0, 1, 2, \dots, \\ 0 &= \mathcal{W}_{(L)}^{sk}(B^0, B^1, \dots, B^{k+s-L}), \quad 0 \leq s < L, \quad k \geq L-s. \\ \mathcal{W}_{(L)}^{kn} &= \bar{\mathcal{W}}_{(L)}^{kn} = \text{const}, \quad k = 0, 1, 2, \dots, L-1, \quad 0 \leq n \leq L-1-k. \end{aligned} \quad (29)$$

Thus, an arbitrary hydrodynamic chain determined by such L -bracket has at least $L+2$ conservation laws (for an arbitrary Hamiltonian density \mathbf{H}_{L+1}), where the first L conservation laws of the Casimirs are

$$B_t^k = \partial_x \left(\sum_{n=0}^{L-k-1} (\bar{\mathcal{W}}_{(L)}^{kn} + \bar{\mathcal{W}}_{(L)}^{nk}) \frac{\partial \mathbf{H}_{L+1}}{\partial B^n} + \sum_{n=L-k}^{L+1} \mathcal{W}_{(L)}^{nk} \frac{\partial \mathbf{H}_{L+1}}{\partial B^n} \right), \quad k = 0, 1, 2, \dots, L-1.$$

The conservation law of the momentum is

$$B_t^L = \partial_x \left(\sum_{n=0}^{L+1} (\mathcal{W}_{(L)}^{nL} + B^n) \frac{\partial \mathbf{H}_{L+1}}{\partial B^n} - \mathbf{H}_{L+1} \right).$$

The conservation law of the energy is

$$\partial_t \mathbf{H}_{L+1} = \partial_x \left[\sum_{k=0}^{L+1} \sum_{n=0}^{L+1} \mathcal{W}_{(L)}^{kn} \frac{\partial \mathbf{H}_{L+1}}{\partial B^k} \frac{\partial \mathbf{H}_{L+1}}{\partial B^n} \right].$$

By the construction all Hamiltonian structures for the Kupershmidt hydrodynamic chain are written in the Liouville coordinates (see [7], [19]). Thus, all Hamiltonian structures presented below can be easily verified by a substitution in above formulas.

Let us re-compute the canonical Poisson bracket (nonzero components are written below only; see [7])

$$\{b^i(x), b^i(x')\} = (\varepsilon_i)^{-1} \delta'(x - x') \quad (30)$$

via the moments $B_{(\gamma)}^k$ (see (2)). Then the r.h.s. of the identity

$$\{B_{(\gamma)}^k, B_{(\gamma)}^n\} = \beta^2 \sum_{n=1}^N \varepsilon_i [(b^i)^{\beta(k+n+2\gamma)-2} \delta'(x - x') + (\beta(n + \gamma) - 1)(b^i)^{\beta(k+n+2\gamma)-3} b_x^i \delta'(x - x')]]$$

can be expressed via the moment $B_{(\gamma)}^m$ only, iff $\gamma = 2/\beta - L$, where L is an integer. Thus, we have

$$\{B_{(2/\beta-L)}^k, B_{(2/\beta-L)}^n\} = \beta^2 [(k - L + 1/\beta) B_{(2/\beta-L)}^{k+n-L} \partial_x + (n - L + 1/\beta) \partial_x B_{(2/\beta-L)}^{k+n-L}] \delta(x - x').$$

Since the numeration of the moments $B_{(2/\beta-L)}^k$ starts from the origin, then the above formula is valid if $k + n \geq L$. By this reason, this family of the Poisson brackets will be slightly deformed on an extra constant block (see (29)). Let us introduce the new notation $C_{(L+1)}^k = B_{(2/\beta-L)}^k / \beta$. Then the above main part ($k + n \geq L$) of the Poisson brackets presented below (see (28))

$$\{C_{(L+1)}^k, C_{(L+1)}^n\} = [(\beta(k - L) + 1) C_{(L+1)}^{k+n-L} \partial_x + (\beta(n - L) + 1) \partial_x C_{(L+1)}^{k+n-L}] \delta(x - x')$$

is determined by the moments

$$C_{(L+1)}^k = \frac{1}{\beta(k - L) + 2} \sum_{i=1}^M \varepsilon_i (b^i)^{\beta(k-L)+2}, \quad k = 0, 1, 2, \dots \quad (31)$$

and equivalent the Kupershmidt Poisson bracket (28) under the re-numeration $C_{(L+1)}^k \rightarrow C_{(1)}^{k-L}$.

Let us consider $K + M$ component hydrodynamic type system (23)

$$\begin{aligned} \partial_t C_{(K+1)}^k &= \partial_x C_{(K+1)}^{k+1} + C^0 \partial_x C_{(K+1)}^k + [2 + \beta(k - K)] C_{(K+1)}^k C_x^0, \quad k = 0, 1, \dots, K - 2, \\ \partial_t C_{(K+1)}^{K-1} &= \frac{1}{2} \partial_x \left(\sum \varepsilon_i (b^i)^2 \right) + C^0 \partial_x C_{(K+1)}^{K-1} + (2 - \beta) C_{(K+1)}^{K-1} C_x^0, \\ b_t^k &= \partial_x \left(\frac{(b^k)^{\beta+1}}{\beta + 1} + C^0 b^k \right), \quad k = 1, 2, \dots, M \end{aligned} \quad (32)$$

connected with the equation of the Riemann surface (24)

$$\lambda = p^{\beta(K+1)-2 + (K+1 - \frac{2}{\beta})} \left[\beta \sum_{k=0}^{K-1} C_{(K+1)}^k p^{\beta(K-k)-2} + \sum_{m=1}^M \varepsilon_m \left(\frac{b^m}{p} \right)^2 F \left(1, \frac{2}{\beta}, 1 + \frac{2}{\beta}, \left(\frac{b^m}{p} \right)^\beta \right) \right].$$

Since λ satisfies the *linear* equation (5), one can replace λ on an arbitrary function $\tilde{\lambda}(\lambda)$. Let us re-scale $\lambda^{\beta(K+1)-2} \rightarrow \lambda^\beta$.

The above hydrodynamic type system (32) can be written in the Hamiltonian form

$$b_t^i = \frac{1}{\varepsilon_i(\beta+1)} \partial_x \frac{\partial \mathbf{h}_{K+1}}{\partial b^i}, \quad h_t^k = \frac{1}{\beta+1} \partial_x \frac{\partial \mathbf{h}_{K+1}}{\partial h^{K-1-k}}, \quad (33)$$

where the first K coefficients $h^k(\mathbf{C})$ of the B rmann–Lagrange series (see (4), (6) and (11))

$$1 - p\lambda^{-1/\beta} = \sum_{n=0}^{\infty} h^n \lambda^{-n}, \quad \lambda \rightarrow \infty$$

can be obtained from the above equation of the Riemann surface (see [18]). Here $\mathbf{h}_K = \Sigma \varepsilon_i (b^i)^2/2 + \Sigma h^k h^{K-1-k}/2$ is a momentum density, the Hamiltonian density is \mathbf{h}_{K+1} .

Theorem 2: *The Kupershmidt hydrodynamic chain (1) has infinitely many local Hamiltonian structures. The hydrodynamic type system (33) is the Hamiltonian hydrodynamic reduction of the Kupershmidt hydrodynamic chain written in the Liouville coordinates $C_{(K+1)}^k$ of K th local Hamiltonian structure.*

Proof contains two steps:

1. It is necessary to prove that, indeed,

$$\mathbf{h}_K = \frac{1}{2} \sum_{i=0}^M \varepsilon_i (b^i)^2 + \frac{1}{2} \sum_{i=0}^K h^i h^{K-1-i}. \quad (34)$$

The existence of the quadratic relationship between one conservation law density \mathbf{h}_K and the conservation law densities b^i, h^k means (see [26]) the existence of the local Hamiltonian structure (33). Then the Hamiltonian density \mathbf{h}_{K+1} can be found in quadratures.

2. Introducing the moments $C_{(K+1)}^k$ via the moment decomposition (31), the Hamiltonian hydrodynamic type system (32) transforms to the Kupershmidt hydrodynamic chain (12)

$$\partial_t C_{(K+1)}^k = \partial_x C_{(K+1)}^{k+1} + C^0 \partial_x C_{(K+1)}^k + [\beta(k-K) + 2] C_{(K+1)}^k C_x^0, \quad k = 0, 1, 2, \dots; \quad (35)$$

simultaneously, the Hamiltonian structure of the hydrodynamic type system (33) transforms to the K th local Hamiltonian structure of the Kupershmidt hydrodynamic chain. To avoid a complexity of computations in general case, we restrict our consideration on the second and third local Hamiltonian structures, while the first local Hamiltonian structure was established in [14].

The first local Hamiltonian structure (see (28))

$$\partial_t C_{(1)}^k = \frac{1}{\beta+1} [(\beta k + 1) C_{(1)}^{k+n} \partial_x + (\beta n + 1) \partial_x C_{(1)}^{k+n}] \frac{\delta \bar{\mathbf{H}}_1}{\delta C_{(1)}^n}, \quad (36)$$

of the Kupershmidt hydrodynamic chain

$$\partial_t C_{(1)}^k = \partial_x C_{(1)}^{k+1} + C^0 \partial_x C_{(1)}^k + (\beta k + 2) C_{(1)}^k C_x^0, \quad k = 0, 1, 2, \dots$$

is determined by the Hamiltonian $\bar{\mathbf{H}}_1 = \int [C_{(1)}^1 + (\beta+1)(C^0)^2/2] dx$ and by the momentum $\bar{\mathbf{H}}_0 = \int C^0 dx$.

Since the point transformation $C_{(1)}^k(\mathbf{B})$ is invertible (see (14)), then the Poisson bracket $\{C_{(1)}^k, C_{(1)}^n\}$ can be expressed via the moments B^k (see (10)). Thus, this local Poisson bracket $\{B^k, B^n\}_1$ determines the **first** local Hamiltonian structure for the Kupershmidt hydrodynamic chain for an arbitrary index β .

Indeed, since the moment decomposition (31)

$$C_{(1)}^k = \frac{1}{\beta k + 2} \sum_{i=1}^M \varepsilon_i (b^i)^{\beta k + 2}, \quad k = 0, 1, 2, \dots$$

yields the momentum density ($K = 0$)

$$\mathbf{h}_0 = C^0 = \frac{1}{2} \sum_{m=1}^M \varepsilon_m (b^m)^2,$$

then one should just re-compute the canonical Poisson bracket (30) via the moments $C_{(1)}^k$ and check the Jacobi identity (27). Then the Hamiltonian density \mathbf{h}_1 of the hydrodynamic reductions (17) (see (33), $K = 0$)

$$b_t^i = \frac{1}{\varepsilon_i(\beta + 1)} \partial_x \frac{\partial \mathbf{h}_1}{\partial b^i}$$

becomes back the Hamiltonian density \mathbf{H}_1 of the first Hamiltonian structure of the Kupershmidt hydrodynamic chain.

All higher commuting flows (see (36))

$$\partial_t^m C_{(1)}^k = \frac{1}{\beta m + 1} [(\beta k + 1) C_{(1)}^{k+n} \partial_x + (\beta n + 1) \partial_x C_{(1)}^{k+n}] \frac{\delta \bar{\mathbf{H}}_m}{\delta C_{(1)}^n}, \quad m = 2, 3, \dots \quad (37)$$

are determined by the higher Hamiltonians $\bar{\mathbf{H}}_m = \int \mathbf{H}_m(C_{(1)}^0, C_{(1)}^1, \dots, C_{(1)}^m) dx$.

The second local Hamiltonian structure

$$C_t^0 = \frac{1}{\beta + 1} \partial_x \frac{\delta \bar{\mathbf{H}}_2}{\delta C^0},$$

$$\partial_t C_{(2)}^k = \frac{1}{\beta + 1} [(\beta k + 1 - \beta) C_{(2)}^{k+n-1} \partial_x + (\beta n + 1 - \beta) \partial_x C_{(2)}^{k+n-1}] \frac{\delta \bar{\mathbf{H}}_2}{\delta C_{(2)}^n}, \quad k, n \geq 1$$

of the Kupershmidt hydrodynamic chain

$$\partial_t C_{(2)}^k = \partial_x C_{(2)}^{k+1} + C^0 \partial_x C_{(2)}^k + (\beta(k-1) + 2) C_{(2)}^k C_x^0, \quad k = 0, 1, 2, \dots$$

is determined by the Hamiltonian $\bar{\mathbf{H}}_2 = \int [C_{(2)}^2 + (\beta + 1) C^0 C_{(2)}^1 + (3 - \beta)(\beta + 1)(C^0)^3/6] dx$, where the momentum is $\bar{\mathbf{H}}_1 = \int (C_{(2)}^1 + (C^0)^2/2) dx$, the Casimir (annihilator of corresponding Poisson bracket) is $\bar{\mathbf{H}}_0 = \int C^0 dx$.

Since the moment decomposition (31)

$$C_{(2)}^{k+1} = \frac{1}{\beta k + 2} \sum_{i=1}^M \varepsilon_i (b^i)^{\beta k + 2}, \quad k = 0, 1, 2, \dots$$

yields the momentum density ($K = 1$)

$$\mathbf{h}_1 = C_{(2)}^1 + \frac{1}{2}(C^0)^2 \equiv \frac{1}{2} \sum_{m=1}^M \varepsilon_m (b^m)^2 + \frac{1}{2}(h^0)^2,$$

where $h^0 = C^0$, then one should just re-compute the canonical Poisson bracket (see (30)); nonzero components are written below only)

$$\{b^i(x), b^i(x')\} = \delta'(x - x'), \quad \{h^0(x), h^0(x')\} = \delta'(x - x')$$

via the moments $C_{(2)}^k$ and check the Jacobi identity (27). Then the Hamiltonian density \mathbf{h}_2 of the hydrodynamic reductions (33)

$$b_t^i = \partial_x \left(\frac{(b^i)^{\beta+1}}{\beta+1} + h^0 b^i \right), \quad h_t^0 = \partial_x \left(\frac{1}{2} \sum \varepsilon_n (b^n)^2 + \frac{3-\beta}{2} (h^0)^2 \right)$$

written in the Hamiltonian form

$$b_t^i = \frac{1}{\varepsilon_i(\beta+1)} \partial_x \frac{\partial \mathbf{h}_2}{\partial b^i}, \quad h_t^0 = \frac{1}{\beta+1} \partial_x \frac{\partial \mathbf{h}_2}{\partial h^0}$$

becomes back the Hamiltonian density \mathbf{H}_2 of the second Hamiltonian structure of the Kupershmidt hydrodynamic chain.

Since the point transformation $C_{(2)}^k(\mathbf{B})$ is invertible (see (14)), then the Poisson bracket $\{C_{(2)}^k, C_{(2)}^n\}$ can be expressed via the moments B^k . Thus, this local Poisson bracket $\{B^k, B^n\}_2$ determines the **second** local Hamiltonian structure for the Kupershmidt hydrodynamic chain for an arbitrary index β .

The *third* local Hamiltonian structure.

The Kupershmidt hydrodynamic chain

$$\partial_t C_{(3)}^k = \partial_x C_{(3)}^{k+1} + C^0 \partial_x C_{(3)}^k + (\beta(k-2) + 2) C_{(3)}^k C_x^0, \quad k = 0, 1, 2, \dots$$

has the local Hamiltonian structure

$$\begin{aligned} C_t^0 &= \frac{1}{\beta+1} \partial_x \frac{\delta \bar{\mathbf{H}}_3}{\delta C_{(3)}^1}, \quad \partial_t C_{(3)}^1 = \frac{1}{\beta+1} \partial_x \frac{\delta \bar{\mathbf{H}}_3}{\delta C^0} + \frac{\beta-1}{\beta+1} (C^0 \partial_x + \partial_x C^0) \frac{\delta \bar{\mathbf{H}}_3}{\delta C_{(3)}^1}, \\ \partial_t C_{(3)}^k &= \frac{1}{\beta+1} [(\beta k + 1 - 2\beta) C_{(3)}^{k+n-2} \partial_x + (\beta n + 1 - 2\beta) \partial_x C_{(3)}^{k+n-2}] \frac{\delta \bar{\mathbf{H}}_3}{\delta C_{(3)}^n}, \quad k, n \geq 2, \end{aligned}$$

where the momentum is $\bar{\mathbf{H}}_2 = \int [C_{(3)}^2 + C^0 C_{(3)}^1 + (1-\beta)(C^0)^3/2] dx$, two Casimirs are $\bar{\mathbf{H}}_1 = \int [C_{(3)}^1 + (1-\beta)(C^0)^2/2] dx$ and $\bar{\mathbf{H}}_0 = \int C^0 dx$; the Hamiltonian is $\bar{\mathbf{H}}_3 = \int [C_{(3)}^3 + (\beta+1)C^0 C_{(3)}^2 + \frac{\beta+1}{2}(C_{(3)}^1)^2 + \frac{(3-2\beta)(\beta+1)}{2}(C^0)^2 C_{(3)}^1 + \frac{5(2\beta-3)(\beta^2-1)}{24}(C^0)^4] dx$.

Since the moment decomposition (31)

$$C_{(2)}^{k+2} = \frac{1}{\beta k + 2} \sum_{i=1}^M \varepsilon_i (b^i)^{\beta k + 2}, \quad k = 0, 1, 2, \dots$$

yields the momentum density ($K = 2$)

$$\mathbf{h}_2 = C_{(3)}^2 + C^0 C_{(3)}^1 + \frac{1-\beta}{2}(C^0)^3 \equiv \frac{1}{2} \sum_{m=1}^M \varepsilon_m (b^m)^2 + h^0 h^1,$$

where $h^0 = C^0$, $h^1 = C_{(3)}^1 + (1-\beta)(C^0)^2/2$, then one should just re-compute the canonical Poisson bracket (see (30)); nonzero components are written below only

$$\{b^i(x), b^i(x')\} = \delta'(x - x'), \quad \{h^0(x), h^1(x')\} = \{h^1(x), h^0(x')\} = \delta'(x - x')$$

via the moments $C_{(2)}^k$ and check the Jacobi identity (27). Then the Hamiltonian density \mathbf{h}_3 of the hydrodynamic reductions (33)

$$\begin{aligned} b_t^i &= \partial_x \left(\frac{(b^i)^{\beta+1}}{\beta+1} + h^0 b^i \right), \quad h_t^0 = \partial_x \left[h^1 + \frac{2-\beta}{2} (h^0)^2 \right], \\ h_t^1 &= \partial_x \left(\frac{1}{2} \sum \varepsilon_n (b^n)^2 + (2-\beta) h^0 h^1 + \frac{\beta(\beta-1)}{6} (h^0)^3 \right) \end{aligned}$$

written in the Hamiltonian form

$$b_t^i = \frac{1}{\varepsilon_i(\beta+1)} \partial_x \frac{\partial \mathbf{h}_3}{\partial b^i}, \quad h_t^0 = \frac{1}{\beta+1} \partial_x \frac{\partial \mathbf{h}_3}{\partial h^1}, \quad h_t^1 = \frac{1}{\beta+1} \partial_x \frac{\partial \mathbf{h}_3}{\partial h^0}$$

becomes back the Hamiltonian density \mathbf{H}_3 of the second Hamiltonian structure of the Kupershmidt hydrodynamic chain.

Since the point transformation $C_{(3)}^k(\mathbf{B})$ is invertible (see (14)), then the Poisson bracket $\{C_{(3)}^k, C_{(3)}^n\}$ can be expressed via the moments B^k . Thus, this local Poisson bracket $\{B^k, B^n\}_3$ determines the **third** local Hamiltonian structure for the Kupershmidt hydrodynamic chain for an arbitrary index β .

In the same way all other local Hamiltonian structures can be constructed. The Kupershmidt hydrodynamic chain (35) is associated with the Poisson K -bracket, where K Casimirs h^k are *homogeneous polynomials* with respect to the first moments $C_{(K+1)}^k$, $k = 0, 1, \dots, K-1$. All other higher moments are functions of the field variables b^k (see (31))

$$C_{(K+1)}^{k+K} = \frac{1}{\beta k + 2} \sum_{i=1}^M \varepsilon_i (b^i)^{\beta k + 2}, \quad k = 0, 1, 2, \dots \quad (38)$$

Corresponding Poisson brackets $\{C_{(K+1)}^k, C_{(K+1)}^n\}$ under the invertible point transformation (14) can be written via the same set of the field variables B^k . These Poisson brackets $\{B^k, B^n\}_{K+1}$ determine the Kupershmidt hydrodynamic chain (10)

$$B_t^k = \frac{1}{\beta+1} \{B^k, \bar{\mathbf{H}}_{K+1}\}_{K+1}.$$

by the Hamiltonians $\bar{\mathbf{H}}_{K+1}$, $K = 0, 1, 2, \dots$, respectively. All higher commuting flows also have the same set of the Hamiltonian structures

$$B_{t^m}^k = \frac{1}{\beta m + 1} \{B^k, \bar{\mathbf{H}}_{K+m}\}_{K+1}, \quad m = 0, 1, 2, \dots$$

All other Hamiltonian structures are **nonlocal** (see [8]). For instance, the simplest such *nonlocal* Hamiltonian structure is associated with the Kupershmidt hydrodynamic chain written in the form (cf. (35))

$$\partial_t C_{(0)}^k = \partial_x C_{(0)}^{k+1} + C_{(0)}^0 \partial_x C_{(0)}^k + (\beta(k+1) + 2) C_{(0)}^k C_x^0, \quad k = 0, 1, 2, \dots \quad (39)$$

Theorem 3: *The Kupershmidt hydrodynamic chain (39) has the nonlocal Hamiltonian structure*

$$\begin{aligned} \partial_t C_{(0)}^k = \{C_{(0)}^k, \bar{\mathbf{H}}_0\} \equiv & \frac{1}{\beta+1} [(\beta k + \beta + 1) C_{(0)}^{k+n+1} \partial_x + (\beta n + \beta + 1) \partial_x C_{(0)}^{k+n+1} \\ & + \frac{1}{\beta} [(\beta k + \beta + 2)(\beta n + \beta + 2) C_{(0)}^k C_{(0)}^n \partial_x + (\beta k + \beta + 2)(\beta n + \beta + 1) C_{(0)}^k (C_{(0)}^n)_x \\ & + (\beta n + \beta + 2) C_{(0)}^n (C_{(0)}^k)_x - (C_{(0)}^k)_x \partial_x^{-1} (C_{(0)}^n)_x] \frac{\delta \bar{\mathbf{H}}_0}{\delta C_{(0)}^n}, \end{aligned}$$

where the Hamiltonian is $\bar{\mathbf{H}}_0 = \int C^0 dx$.

Proof: The Kupershmidt hydrodynamic chain (39) contains the hydrodynamic reduction (17)

$$b_t^i = \partial_x \left(\frac{(b^i)^{\beta+1}}{\beta+1} + C^0 b^i \right), \quad (40)$$

where the moment decomposition is given by (cf. (38))

$$C_{(0)}^{k-1} = \frac{1}{\beta k + 2} \sum_{i=1}^M \varepsilon_i (b^i)^{\beta k + 2}, \quad k = 1, 2, \dots$$

However, while the hydrodynamic reductions (3) considered in all above examples have M independent field variables b^k , this hydrodynamic type system has $M-1$ independent field variables only. Indeed, the above hydrodynamic reduction (40) has the *quadratic* constraint

$$\sum_{n=1}^M \varepsilon_n (b^n)^2 = -1$$

fixing the nonlocal Hamiltonian structure associated with a metric of constant curvature if $\gamma = 2/\beta + 1$ (see details in [10], [25], [26]). Thus, the above hydrodynamic type system can be written in the Hamiltonian form

$$b_t^i = \frac{1}{\beta+1} \partial_x \left[\frac{1}{\varepsilon_i} \frac{\partial \mathbf{h}_0}{\partial b^i} + b^i \left(\sum_{k=1}^{M-1} b^k \frac{\partial \mathbf{h}_0}{\partial b^k} - \mathbf{h}_0 \right) \right], \quad i = 1, 2, \dots, M-1,$$

where $b^M(b^1, b^2, \dots, b^{M-1})$ can be expressed from the above quadratic constraint.

A straightforward re-calculation yields the corresponding nonlocal Hamiltonian structure of the Kupershmidt hydrodynamic chain (39).

This is the first example of nonlocal Hamiltonian structures for hydrodynamic chains. A direct verification of the Jacobi identity is not so simple. However, under special reciprocal transformation this nonlocal Hamiltonian structure becomes local (see [25]). Such reciprocal transformations for nonlocal Hamiltonian structures will be considered elsewhere.

6 The extended Kupershmidt chains

The Kupershmidt hydrodynamic chain (12) has N component hydrodynamic reductions (3) determined by the moment decomposition (2). This moment decomposition is defined for any values of the index k . However, (2) has been used for $k = 0, 1, 2, \dots$ only. Let us remove this restriction.

Definition 2: *The hydrodynamic chain*

$$\partial_t B_{(\gamma)}^k = \partial_x B_{(\gamma)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(\gamma)}^k + (k + \gamma) B_{(\gamma)}^k B_x^0, \quad k \in \mathbf{Z} \quad (41)$$

is said to be the **extended** Kupershmidt hydrodynamic chain.

(41) has the same set of hydrodynamic reductions as (12). Thus, the extended Kupershmidt hydrodynamic chain is a natural generalization of the Kupershmidt hydrodynamic chain.

As well as the hydrodynamic type system (3), the extended Kupershmidt hydrodynamic chain has *negative* conservation laws and *negative* commuting flows. For instance, the couple of the first negative conservation laws is

$$\partial_{t^1} \mathbf{H}_{-1} = \partial_{t^0} \left(\frac{B^0}{\beta} \mathbf{H}_{-1} \right), \quad \partial_{t^1} \mathbf{H}_{-2} = \partial_{t^0} \left(\frac{\beta(\gamma - 1)}{\beta + 1} (\mathbf{H}_{-1})^{\beta+1} + \frac{B^0}{\beta} (\mathbf{H}_{-1})^{\beta(2-\gamma)+1} B_{(\gamma)}^{-2} \right),$$

where $t^1 \equiv t$, $t^0 \equiv x$ and

$$\mathbf{H}_{-1} = \left(1 + (\gamma - 1) B_{(\gamma)}^{-1} \right)^{\frac{1}{\beta(\gamma-1)}}, \quad \mathbf{H}_{-2} = B_{(\gamma)}^{-2} (\mathbf{H}_{-1})^{\beta(2-\gamma)+1}.$$

The extended Kupershmidt hydrodynamic chain (41) and its commuting flows have the same set of local and nonlocal Hamiltonian structures. For instance, the first *negative* commuting flow

$$\partial_{t^{-1}} B_{(\gamma)}^k = (\mathbf{H}_{-1})^{\beta-2} [\mathbf{H}_{-1} \partial_x B_{(\gamma)}^{k-1} - \beta(k + \gamma - 1) B_{(\gamma)}^{k-1} \partial_x \mathbf{H}_{-1}] \quad (42)$$

can be determined by the first local Hamiltonian structure (37)

$$\partial_{t^{-1}} C_{(1)}^k = \frac{1}{1 - \beta} [(\beta k + 1) C_{(1)}^{k-1} \partial_x + (1 - \beta) \partial_x C_{(1)}^{k-1}] \frac{\delta \bar{\mathbf{H}}_{-1}}{\delta C_{(1)}^{-1}}, \quad k \in \mathbf{Z},$$

where the first *negative* Hamiltonian is $\bar{\mathbf{H}}_{-1} = \int \left(1 + (2/\beta - 1) C_{(1)}^{-1} \right)^{\frac{1}{2-\beta}} dx$.

Remark: Let us restrict our consideration of the above hydrodynamic chain (42) on *purely negative* values of the discrete variable k . The corresponding hydrodynamic chain is

$$\partial_y \tilde{B}_{(\gamma)}^k = (\tilde{B}_{(\gamma)}^0 + \varepsilon)^{\frac{\beta-1}{\beta(\gamma-1)-1}} \left[(\tilde{B}_{(\gamma)}^0 + \varepsilon) \partial_x \tilde{B}_{(\gamma)}^{k+1} + \frac{k + 2 - \gamma}{\gamma - 1} \tilde{B}_{(\gamma)}^{k+1} \partial_x \tilde{B}_{(\gamma)}^0 \right], \quad k = 0, 1, 2, \dots,$$

where $y = t^{-1} \varepsilon^{\frac{1-\beta}{\beta(\gamma-1)}}$, $\tilde{B}_{(\gamma)}^k = \varepsilon(\gamma - 1) B_{(\gamma)}^{-k-1}$ and ε is an arbitrary constant removable by the shift $\tilde{B}_{(\gamma)}^0 + \varepsilon \rightarrow \tilde{B}_{(\gamma)}^0$. This hydrodynamic chain was studied in [4], [16], [21], [28] and

[29]. This constant ε plays an important role in a non-degeneracy of the corresponding hydrodynamic reductions (2). If $\varepsilon = 0$, then the extra constraint $\Sigma \varepsilon_k (b^k)^{\beta\gamma} = \text{const}$ appears.

The simplest N parametric family of N component hydrodynamic reductions of the hydrodynamic chain (42)

$$b_{t-1}^i = \partial_x \left[\frac{(\mathbf{H}_{-1}/b^i)^{\beta-1}}{1-\beta} \right] \quad (43)$$

commutes with (3). The compatibility condition $\partial_t(\partial_{t-1}p) = \partial_{t-1}(\partial_t p)$ of the generating functions of conservation laws (4) and (we replace $b^i \rightarrow p$, see [28])

$$p_{t-1} = \partial_x \left[\frac{(p/\mathbf{H}_{-1})^{1-\beta}}{1-\beta} \right] \quad (44)$$

yields (see [21]) 2+1 quasilinear system

$$\partial_{t-1} \mathbf{H}_0 = \partial_x \frac{(\mathbf{H}_{-1})^{\beta-1}}{\beta-1}, \quad \partial_t \mathbf{H}_{-1} = \partial_x (\mathbf{H}_0 \mathbf{H}_{-1}),$$

where $\mathbf{H}_0 = B^0/\beta$.

As well as in the second section a consistency of the generating function of conservation laws (44) with the hydrodynamic type system (43) yields (7). Taking into account the moment decomposition for the negative moments (see (2))

$$B_{(\gamma)}^{-k} = \frac{1}{\gamma-k} \sum_{i=1}^N \varepsilon_i (b^i)^{\beta(\gamma-k)}, \quad \gamma \neq 0, 1, 2, \dots,$$

the **another** equation of the Riemann mapping (cf. (6) and (8))

$$\lambda = q^{1-\gamma} - (1-\gamma) \sum_{k=1}^{\infty} B_{(\gamma)}^{-k} q^{k-\gamma}, \quad \lambda \rightarrow 0 \quad (45)$$

can be found (the above moment decomposition degenerates according to (16) in the exceptional cases $\gamma = 0, 1, 2, \dots$). As well as in the *positive* case (14) the negative moments $B_{(\gamma)}^{-k}$ with the distinct parameters γ can be expressed via each other by the invertible transformations

$$B_{(\gamma)}^{-1} = \frac{1 - (1 - B^{-1})^{1-\gamma}}{1-\gamma}, \quad B_{(\gamma)}^{-2} = B^{-2} (1 - B^{-1})^{-\gamma}, \dots$$

If $\gamma = 1$, the extended Kupershmidt hydrodynamic chain (41)

$$\partial_t B_{(1)}^k = \partial_x B_{(1)}^{k+1} + \frac{1}{\beta} B^0 \partial_x B_{(1)}^k + (k+1) B_{(1)}^k B_x^0, \quad k \in \mathbf{Z}$$

has the first negative conservation law

$$\partial_t \mathbf{H}_{-1} = \frac{1}{\beta} \partial_x (B^0 \mathbf{H}_{-1}),$$

where $\mathbf{H}_{-1} = e^{B_{(1)}^{-1}/\beta}$, and the first negative commuting flow

$$\partial_{t^{-1}} B_{(1)}^k = e^{\frac{\beta-1}{\beta} B_{(1)}^{-1}} [\partial_x B_{(1)}^{k-1} - k B_{(1)}^{k-1} \partial_x B_{(1)}^{-1}].$$

The negative moments $B_{(1)}^{-k}$ are connected with the equation of the Riemann mapping (45) reduced to (cf. (15))

$$\lambda = \ln q - \sum_{k=1}^{\infty} B_{(1)}^{-k} q^{k-1} \quad \Leftrightarrow \quad \lambda = \frac{1}{q} \exp \left(\sum_{k=1}^{\infty} B_{(1)}^{-k} q^{k-1} \right), \quad \lambda \rightarrow \infty, \quad q \rightarrow 0.$$

Since $B_{(1)}^{-1} = \beta \Sigma \varepsilon_k \ln b^k$ and $\Sigma \varepsilon_k = 0$ in this exceptional case ($\gamma = 1$, see (16)), then $\mathbf{H}_{-1} = \Pi(b^k)^{\varepsilon_k}$.

Summary: The Kupershmidt hydrodynamic chain (12) can be extended on the negative moments $B_{(\gamma)}^{-k}$. The extended Kupershmidt hydrodynamic chain (41) has infinitely many higher (positive) and lower (negative) conservation laws and commuting flows. Two different asymptotic Riemann mappings (6) and (45) are connected with positive and negative parts of these hydrodynamic chains, respectively.

Let us associate the “time” indexes m of commuting flows $\partial_{t^m} B_{(\gamma)}^k$ and the moment indexes k with the corresponding dots (m, k) on the plane. By this reason the hierarchy of the Kupershmidt hydrodynamic chain (including all conservation laws enumerated by the index k and all commuting flows enumerated by the index m) we call the Kupershmidt hydrodynamic *lattice*.

7 Linear transformation of independent variables

In this section we describe the simplest discrete symmetry of the Kupershmidt hydrodynamic lattice (see the corresponding transformation of hydrodynamic reductions in [28]). By this reason, the generating function of conservation law densities $p^{(\beta)}$, all conservation law densities $\mathbf{H}_k^{(\beta)}$ and all moments $B_{(\beta, \gamma)}^k$ we write with the extra index β .

Theorem 4: *The generating function of conservation laws (44)*

$$p_{t^{-1}}^{(\beta)} = \partial_{t^0} \left[\frac{\left(p^{(\beta)} / \mathbf{H}_{-1}^{(\beta)} \right)^{1-\beta}}{1-\beta} \right]$$

is invariant under the transformation $t^0 \leftrightarrow t^{-1}$. Then

$$p^{(\tilde{\beta})} = \left(p^{(\beta)} / \mathbf{H}_{-1}^{(\beta)} \right)^{1-\beta}, \quad \mathbf{H}_{-1}^{(\tilde{\beta})} = \left(\mathbf{H}_{-1}^{(\beta)} \right)^{\beta-1}, \quad \tilde{B}_{(\tilde{\beta}, \tilde{\gamma})}^k = B_{(\beta, \gamma)}^{-k-2} \left(\mathbf{H}_{-1}^{(\beta)} \right)^{\beta(k+2-\gamma)},$$

where

$$\tilde{\beta} = \frac{\beta}{\beta-1}, \quad \tilde{\gamma} = 2-\gamma$$

and $\tilde{\varepsilon}_i = -\varepsilon_i$ in the corresponding moment decompositions (2).

Proof: can be obtained by a straightforward substitution in (42) and (44).

Corollary: This transformation connects (6) with (45). These equations of the Riemann mapping are related by

$$q^{(\tilde{\beta})} = \frac{(\mathbf{H}_{-1}^{(\beta)})^\beta}{q^{(\beta)}}.$$

Remark: If $\gamma = 1$, then $\tilde{\gamma} = 1$ and $\tilde{B}_{(\tilde{\beta},1)}^k = B_{(\beta,1)}^{-k-2} \exp[(k+1)B_{(\beta,1)}^{-1}]$.

Below we extend this discrete transformation $t^0 \leftrightarrow t^{-1}$ on all other “times” $t^{k-1} \leftrightarrow t^{-k}$. Following the recipe given in [30] (see also [28]) one can seek a *generating function of conservation laws and commuting flows* in the form

$$\partial_{\tau(\zeta)} p(\lambda) = \partial_{t^0} F(p(\lambda), p(\zeta)), \quad (46)$$

where $\partial_{\tau(\zeta)}$ is the so-called “vertex” operator (see, for instance, [5] and details below). The compatibility conditions $\partial_{t^1}(\partial_{\tau(\zeta)} p(\lambda)) = \partial_{\tau(\zeta)}(\partial_{t^1} p(\lambda))$ and $\partial_{t^{-1}}(\partial_{\tau(\zeta)} p(\lambda)) = \partial_{\tau(\zeta)}(\partial_{t^{-1}} p(\lambda))$ (see (4) and (44)) yield

$$\partial_{\tau(\zeta)} \mathbf{H}_0 = \partial_{t^0} \frac{p^{\beta-1}(\zeta)}{\beta-1}, \quad \partial_{\tau(\zeta)} \mathbf{H}_{-1} = -\partial_{t^0} \frac{\mathbf{H}_{-1}}{p(\zeta)}, \quad (47)$$

where

$$dF(w) = \frac{dw}{w^\beta - 1}, \quad w = p(\lambda)/p(\zeta).$$

Taking into account two distinct Riemann mappings (6) and (45) one can introduce the inverse asymptotics $p(\zeta)$ and consistent formal series for the vertex operator $\partial_{\tau(\zeta)}$

$$\begin{aligned} \partial_{\tau(\zeta)} &= -\zeta^{-1/\beta} \sum_{k=0}^{\infty} \zeta^{-k} \partial_{t^k}, & p(\zeta) &= \zeta^{1/\beta} \left(1 - \sum_{k=1}^{\infty} \zeta^{-k} \mathbf{H}_{k-1} \right), \\ \partial_{\tau(\zeta)} &= \zeta^{1/\beta} \sum_{k=1}^{\infty} \zeta^{-k} \partial_{t^{-k}}, & p(\zeta) &= \zeta^{-1/\beta} \sum_{k=0}^{\infty} \zeta^{-k} \mathbf{H}_{-k-1}, \end{aligned} \quad (48)$$

when $\zeta \rightarrow \infty$.

Applying above series to *the generating function of conservation laws and commuting flows* (see (46)) one can obtain infinite series of separate generating functions of conservation laws and commuting flows (cf. (4), (44), (47))

$$\begin{aligned} \partial_{t^1} p &= \partial_{t^0} \left(\frac{p^{\beta+1}}{\beta+1} + \mathbf{H}_0 p \right), & \partial_{t^2} p &= \partial_{t^0} \left(\frac{p^{2\beta+1}}{2\beta+1} + \mathbf{H}_0 p^{\beta+1} + (\mathbf{H}_1 + (\mathbf{H}_0)^2) p \right), \dots \\ \partial_{t^{-1}} p &= \partial_{t^0} \left(\frac{(p/\mathbf{H}_{-1})^{1-\beta}}{1-\beta} \right), & \partial_{t^{-2}} p &= \partial_{t^0} \left(\frac{(p/\mathbf{H}_{-1})^{1-2\beta}}{1-2\beta} - \frac{\mathbf{H}_{-3}}{\mathbf{H}_{-2}} (p/\mathbf{H}_{-1})^{1-\beta} \right), \dots \\ \partial_{\tau} \mathbf{H}_1 &= \partial_{t^0} \left(\frac{p^{2\beta-1}}{2\beta-1} + \mathbf{H}_0 p^{\beta-1} \right), & \partial_{\tau} \mathbf{H}_2 &= \partial_{t^0} \left(\frac{p^{3\beta-1}}{3\beta-1} + \mathbf{H}_0 p^{2\beta-1} + (\mathbf{H}_1 + \frac{\beta}{2} (\mathbf{H}_0)^2) p^{\beta-1} \right), \dots \\ \partial_{\tau} \mathbf{H}_0 &= \partial_{t^0} \frac{p^{\beta-1}}{\beta-1}, & \partial_{\tau} \mathbf{H}_{-1} &= -\partial_{t^0} \frac{\mathbf{H}_{-1}}{p}, & \partial_{\tau} \mathbf{H}_{-2} &= -\partial_{t^0} \left(\frac{\mathbf{H}_{-2}}{p} + \frac{(\mathbf{H}_{-1}/p)^{\beta+1}}{\beta+1} \right), \dots \end{aligned}$$

Let us rewrite the generating function of conservation laws of the Kupershmidt hydrodynamic chain together with the second line of above formulas in the potential form

$$d\xi = \dots + \left(\frac{p^{\beta+1}}{\beta+1} + \mathbf{H}_0 p \right) dt^1 + p dt^0 + \frac{(p/\mathbf{H}_{-1})^{1-\beta}}{1-\beta} dt^{-1} + \left(\frac{(p/\mathbf{H}_{-1})^{1-2\beta}}{1-2\beta} - \frac{\mathbf{H}_{-3}}{\mathbf{H}_{-2}} (p/\mathbf{H}_{-1})^{1-\beta} \right) dt^{-2} + \dots$$

The compatibility conditions $(\xi_{t-1})_{t^0} = (\xi_{t^0})_{t-1}$ and $(\xi_{t-1})_{t^{-2}} = (\xi_{t^{-2}})_{t-1}$ yield the generating functions of conservation laws (cf. (4) and (44))

$$\partial_{t^0} p^{(\tilde{\beta})} = \partial_{t^{-1}} \left[\frac{(p^{(\tilde{\beta})}/\mathbf{H}_{-1}^{(\tilde{\beta})})^{1-\tilde{\beta}}}{1-\tilde{\beta}} \right], \quad \partial_{t^{-2}} p^{(\tilde{\beta})} = \partial_{t^{-1}} \left[\frac{(p^{(\tilde{\beta})})^{\tilde{\beta}+1}}{\tilde{\beta}+1} + \mathbf{H}_0^{(\tilde{\beta})} p^{(\tilde{\beta})} \right],$$

where (see the above theorem)

$$\tilde{\beta} = \frac{\beta}{\beta-1}, \quad p^{(\tilde{\beta})} = \left(\frac{p^{(\beta)}}{\mathbf{H}_{-1}^{(\beta)}} \right)^{1-\beta}, \quad \mathbf{H}_{-1}^{(\tilde{\beta})} = \left(\mathbf{H}_{-1}^{(\beta)} \right)^{\beta-1}, \quad \mathbf{H}_0^{(\tilde{\beta})} = (\beta-1) \frac{\mathbf{H}_{-2}^{(\beta)}}{\mathbf{H}_{-1}^{(\beta)}}.$$

The substitution of the third and the fourth series (48) in the second above equation

$$p^{(\tilde{\beta})} = \left(\frac{p^{(\beta)}}{\mathbf{H}_{-1}^{(\beta)}} \right)^{1-\beta}$$

yields explicit expressions $\mathbf{H}_k^{(\tilde{\beta})}$ via $\mathbf{H}_n^{(\beta)}$ (this is an invertible transformation) for any indexes k . For instance,

$$\begin{aligned} \tilde{\mathbf{H}}_1 &= (\beta-1) \left(\frac{\mathbf{H}_{-3}}{\mathbf{H}_{-1}} - \frac{\beta}{2} \frac{(\mathbf{H}_{-2})^2}{(\mathbf{H}_{-1})^2} \right), \quad \tilde{\mathbf{H}}_2 = (\beta-1) \left(\frac{\mathbf{H}_{-4}}{\mathbf{H}_{-1}} - \beta \frac{\mathbf{H}_{-2}\mathbf{H}_{-3}}{(\mathbf{H}_{-1})^2} + \frac{\beta(\beta+1)}{6} \frac{(\mathbf{H}_{-2})^3}{(\mathbf{H}_{-1})^3} \right), \dots, \\ \tilde{\mathbf{H}}_{-2} &= (\beta-1) \mathbf{H}_0 (\mathbf{H}_{-1})^{\beta-1}, \quad \tilde{\mathbf{H}}_{-3} = (\beta-1) (\mathbf{H}_{-1})^{\beta-1} \left(\mathbf{H}_1 + \frac{\beta(\beta+1)}{2} (\mathbf{H}_0)^2 \right), \dots, \end{aligned}$$

where for simplicity we use the temporary notation $\mathbf{H}_k = \mathbf{H}_k^{(\beta)}$, $\tilde{\mathbf{H}}_k = \mathbf{H}_k^{(\tilde{\beta})}$.

Thus, the Kupershmidt hydrodynamic *lattice* (i.e. a whole family of commuting hydrodynamic *chains*) admits the transformation $\beta \leftrightarrow \tilde{\beta}$, $t^k \leftrightarrow \tilde{t}^{-1-k}$, $k = 0, \pm 1, \pm 2, \dots$. It means that the Kupershmidt hydrodynamic chain with the index β has infinitely many *negative* conservation laws, which are *positive* conservation laws for the Kupershmidt hydrodynamic chain with the index $\tilde{\beta}$. For instance, the modified Benney chain ($\beta = 1$) is *matched together* with the dispersionless limit of the discrete KP hierarchy ($\beta = \infty$). The case $\beta = 2$ (dispersionless limit of the Veselov–Novikov equation) is *invariant* under the above transformation.

8 Reciprocal transformations

In the theory of integrable dispersive and dispersionless systems the concept of *reciprocal transformations* was introduced by S.A. Chaplygin (see details in [33] and [34]). Also

reciprocal transformations are useful in an application to hydrodynamic chains (see first examples in [9] and [23]). Let us substitute the Taylor series (48) in the generating function of commuting flows

$$\partial_{\tau(\zeta)} \mathbf{H}_{-1} = -\partial_{t^0} \frac{\mathbf{H}_{-1}}{p(\zeta)}$$

and rewrite corresponding conservation laws in the potential form

$$dy^0 = \dots + \left[\frac{\mathbf{H}_{-3}}{\mathbf{H}_{-1}} - \frac{(\mathbf{H}_{-2})^2}{(\mathbf{H}_{-1})^2} \right] dt^{-2} + \frac{\mathbf{H}_{-2}}{\mathbf{H}_{-1}} dt^{-1} + \mathbf{H}_{-1} [dt^0 + \mathbf{H}_0 dt^1 + (\mathbf{H}_1 + (\mathbf{H}_0)^2) dt^2 + \dots].$$

The reciprocal transformation $dy^0 = \dots$ and $dy^k = dt^{-k}, k = \pm 1, \pm 2, \dots$ applied to the generating functions of conservation laws (see the previous section) yields the generating functions of conservation laws

$$\begin{aligned} \partial_{y^1} \tilde{p} &= \partial_{y^0} \left(\frac{\tilde{p}^{1-\beta}}{1-\beta} + \tilde{\mathbf{H}}_0 \tilde{p} \right), & \partial_{y^2} \tilde{p} &= \partial_{y^0} \left(\frac{\tilde{p}^{1-2\beta}}{1-2\beta} + \tilde{\mathbf{H}}_0 \tilde{p}^{1-\beta} + (\tilde{\mathbf{H}}_1 + (\tilde{\mathbf{H}}_0)^2) \tilde{p} \right), \dots \\ \partial_{y^{-1}} \tilde{p} &= \partial_{y^0} \left(\frac{(\tilde{p}/\tilde{\mathbf{H}}_{-1})^{\beta+1}}{\beta+1} \right), & \partial_{y^{-2}} \tilde{p} &= \partial_{y^0} \left(\frac{(\tilde{p}/\tilde{\mathbf{H}}_{-1})^{2\beta+1}}{2\beta+1} - \frac{\tilde{\mathbf{H}}_{-2}}{\tilde{\mathbf{H}}_{-1}} (\tilde{p}/\tilde{\mathbf{H}}_{-1})^{\beta+1} \right), \dots \end{aligned}$$

where

$$\tilde{p} = \frac{p}{\mathbf{H}_{-1}}, \quad \tilde{\mathbf{H}}_{-1} = \frac{1}{\mathbf{H}_{-1}}, \quad \tilde{\mathbf{H}}_{-2} = -\frac{\mathbf{H}_0}{\mathbf{H}_{-1}}, \quad \tilde{\mathbf{H}}_{-3} = -\frac{\mathbf{H}_1}{\mathbf{H}_{-1}}, \dots, \quad \tilde{\mathbf{H}}_0 = -\frac{\mathbf{H}_{-2}}{\mathbf{H}_{-1}}, \quad \tilde{\mathbf{H}}_1 = -\frac{\mathbf{H}_{-3}}{\mathbf{H}_{-1}}, \dots$$

Theorem 5: *The Kupershmidt hydrodynamic chains (12)*

$$\partial_{t^1} B_{(\beta, \gamma)}^n = \partial_{t^0} B_{(\beta, \gamma)}^{n+1} + \frac{1}{\beta} B^0 \partial_{t^0} B_{(\beta, \gamma)}^n + (n + \gamma) B_{(\beta, \gamma)}^n \partial_{t^0} B^0, \quad n = 0, 1, 2, \dots,$$

$$\partial_{y^{-1}} \tilde{B}_{(\tilde{\beta}, \tilde{\gamma})}^k = (\tilde{\mathbf{H}}_{-1})^{\tilde{\beta}-2} [\tilde{\mathbf{H}}_{-1} \partial_{y^0} \tilde{B}_{(\tilde{\beta}, \tilde{\gamma})}^{k-1} - \tilde{\beta}(k + \tilde{\gamma} - 1) \tilde{B}_{(\tilde{\beta}, \tilde{\gamma})}^{k-1} \partial_{y^0} \tilde{\mathbf{H}}_{-1}], \quad n = 0, 1, 2, \dots$$

are related by the above reciprocal transformation, where (cf. the theorem from the previous section)

$$\tilde{B}_{(\tilde{\beta}, \tilde{\gamma})}^k = B_{(\beta, \gamma)}^{-k-2} \left(\mathbf{H}_{-1}^{(\beta)} \right)^{\beta(k+2-\gamma)}, \quad \tilde{\beta} = -\beta, \quad \tilde{\gamma} = 2 - \gamma$$

and $\tilde{\varepsilon}_i = -\varepsilon_i$ in the corresponding moment decompositions (2).

Proof: can be obtained by a straightforward substitution of the above three formulas in the above hydrodynamic chains.

Remark: The case $\beta = 1/N$ was considered in [4] (see also [21]). The Kupershmidt hydrodynamic chain (the first above) and its **higher** commuting flows in such case are called “ N -dmKP hierarchy”. The Kupershmidt hydrodynamic chain (the second above) and its **lower** commuting flows are called “ N -dDym hierarchy”. We would like to emphasize again here that corresponding hydrodynamic chains are members of a sole hierarchy. For the case $\beta = 1/N$ all together these commuting flows are called “ N -dToda hierarchy” (see [21]).

Let us take into account the *discrete transformation of independent variables* described in the previous section. Then the index β can be changed in a combination of these both transformations

$$\begin{aligned}\beta_{(0)} &\rightarrow \beta_{(1)} = -\beta_{(0)}, & \beta_{(1)} &\rightarrow \beta_{(2)} = \frac{\beta_{(1)}}{\beta_{(1)} - 1}, & \beta_{(2)} &\rightarrow \beta_{(3)} = -\beta_{(2)}, \dots \\ \beta_{(0)} &\rightarrow \beta_{(-1)} = \frac{\beta_{(0)}}{\beta_{(0)} - 1}, & \beta_{(-1)} &\rightarrow \beta_{(-2)} = -\beta_{(-1)}, & \beta_{(-2)} &\rightarrow \beta_{(-3)} = \frac{\beta_{(-2)}}{\beta_{(-2)} - 1}, \dots\end{aligned}$$

Thus,

$$\beta_{(2K)} = -\beta_{(2K+1)} = \frac{\beta_{(0)}}{\beta_{(0)}K + 1}, \quad K = 0, \pm 1, \pm 2, \dots$$

Let us start, for instance, from the dispersionless limit of the discrete DKP hierarchy, where the *initial* hydrodynamic chain is (see, for instance, [13])

$$B_t^n = B_x^{n+1} + nB^n B_x^0, \quad n = 0, \dots$$

It means that $\beta_{(0)} = \infty$. Since

$$\frac{1}{\beta_{(2K)}} = -\frac{1}{\beta_{(2K+1)}} = K + \frac{1}{\beta_{(0)}}, \quad \frac{1}{\beta_{(-2K)}} = -\frac{1}{\beta_{(-2K+1)}} = -K + \frac{1}{\beta_{(0)}},$$

then corresponding hydrodynamic *lattices* contain the Kupershmidt hydrodynamic chains (12)

$$\bar{B}_t^n = \bar{B}_x^{n+1} + M\bar{B}^0 \bar{B}_x^n + n\bar{B}^n \bar{B}_x^0, \quad n = 0, 1, 2, \dots,$$

$$\tilde{B}_t^n = \tilde{B}_x^{n+1} - L\tilde{B}^0 \tilde{B}_x^n + n\tilde{B}^n \tilde{B}_x^0, \quad n = 0, 1, 2, \dots,$$

which are exactly exclusive cases $\beta = 1/N$ found by a dispersionless limit in [4] (see also [21]). Thus, any hydrodynamic reduction and any solution of corresponding 2+1 quasilinear equations (see below) can be recalculated for the above hydrodynamic chains with arbitrary integer indexes $K_1 \leftrightarrow K_2$.

9 Ideal gas dynamics

The first and the last equations in (21) can be written in the conservative form

$$B_t^0 = \partial_x [B_{(\gamma)}^1 + \frac{1 + \beta\gamma}{2\beta} (B^0)^2], \quad \partial_t \left[\left(B_{(\gamma)}^{N-1} \right)^{\frac{1}{\beta(N-1+\gamma)}} \right] = \partial_x \left[\frac{B^0}{\beta} \left(B_{(\gamma)}^{N-1} \right)^{\frac{1}{\beta(N-1+\gamma)}} \right].$$

The ideal gas dynamics written in physical variables (see, for instance, [22])

$$u_t = \partial_x \left(\frac{u^2}{2} + \frac{v^\beta}{\beta} \right), \quad v_t = \partial_x (uv) \quad (49)$$

is a two component hydrodynamic reduction (see details in [17]) of the Kupershmidt hydrodynamic chain (21), where $\gamma = 0$, $B^0 = \beta u$ and $B^1 = v^\beta$.

Transformations $\beta_{(k)} \rightarrow \beta_{(k+1)}$ described in the two previous sections are compatible with any hydrodynamic reductions of the Kupershmidt hydrodynamic chains (see, for instance, [28]). In this paper we consider these transformations for the ideal gas dynamics (49)

$$\partial_{t_{(0)}^1} v_{(0)} = \partial_{t_{(0)}^0} (u_{(0)} v_{(0)}), \quad \partial_{t_{(0)}^1} u_{(0)} = \partial_{t_{(0)}^0} \left(\frac{u_{(0)}^2}{2} + \frac{v_{(0)}^{\beta_{(0)}}}{\beta_{(0)}} \right).$$

Its commuting flow (see, for instance, [22])

$$\partial_{t_{(0)}^{-1}} v_{(0)} = \partial_{t_{(0)}^0} u_{(0)}, \quad \partial_{t_{(0)}^{-1}} u_{(0)} = \partial_{t_{(0)}^0} \frac{v_{(0)}^{\beta_{(0)}-1}}{\beta_{(0)} - 1},$$

is known as the “nonlinear elasticity equation”. Let us apply the reciprocal transformation described in the previous section. This is nothing but a transition from the Euler to the Lagrangian coordinates. So, the *first iteration* is

$$\begin{aligned} \partial_{t_{(1)}^{-1}} v_{(1)} &= \partial_{t_{(1)}^0} u_{(1)}, & \partial_{t_{(1)}^1} v_{(1)} &= \partial_{t_{(1)}^0} (u_{(1)} v_{(1)}), \\ \partial_{t_{(1)}^{-1}} u_{(1)} &= \partial_{t_{(1)}^0} \frac{v_{(1)}^{\beta_{(1)}-1}}{\beta_{(1)} - 1}, & \partial_{t_{(1)}^1} u_{(1)} &= \partial_{t_{(1)}^0} \left(\frac{u_{(1)}^2}{2} + \frac{v_{(1)}^{\beta_{(1)}}}{\beta_{(1)}} \right), \end{aligned}$$

where

$$v_{(1)} = \frac{1}{v_{(0)}}, \quad u_{(1)} = -u_{(0)}, \quad \beta_{(1)} = -\beta_{(0)}, \quad t_{(1)}^{-1} = t_{(0)}^1, \quad t_{(1)}^1 = t_{(0)}^{-1}.$$

According to the previous section, the *second iteration* is

$$\begin{aligned} \partial_{t_{(2)}^{-1}} v_{(2)} &= \partial_{t_{(2)}^0} u_{(2)}, & \partial_{t_{(2)}^1} v_{(2)} &= \partial_{t_{(2)}^0} (u_{(2)} v_{(2)}), \\ \partial_{t_{(2)}^{-1}} u_{(2)} &= \partial_{t_{(2)}^0} \frac{v_{(2)}^{\beta_{(2)}-1}}{\beta_{(2)} - 1}, & \partial_{t_{(2)}^1} u_{(2)} &= \partial_{t_{(2)}^0} \left(\frac{u_{(2)}^2}{2} + \frac{v_{(2)}^{\beta_{(2)}}}{\beta_{(2)}} \right), \end{aligned}$$

where

$$v_{(2)} = v_{(1)}^{\beta_{(1)}-1}, \quad u_{(2)} = (\beta_{(1)} - 1)u_{(1)}, \quad \beta_{(2)} = \frac{\beta_{(1)}}{\beta_{(1)} - 1}, \quad t_{(2)}^0 = t_{(1)}^{-1}, \quad t_{(2)}^{-1} = t_{(1)}^0, \quad t_{(2)}^1 = t_{(1)}^{-2}.$$

The ideal gas dynamics (49) and the nonlinear elasticity equation in the Riemann invariants have the form

$$\begin{aligned} r_t^1 &= [r^1 - \varepsilon(r^1 + r^2)]r_x^1, & r_y^1 &= (r^1 - r^2)^{2\varepsilon}r_x^1, \\ r_t^2 &= [r^2 - \varepsilon(r^1 + r^2)]r_x^2, & r_y^2 &= -(r^1 - r^2)^{2\varepsilon}r_x^2, \end{aligned} \tag{50}$$

where

$$u = \frac{1-2\varepsilon}{2}(r^1 + r^2), \quad v = \left(\frac{r^1 - r^2}{4}\right)^{1-2\varepsilon}, \quad \beta = \frac{2}{1-2\varepsilon}.$$

If $\varepsilon = 1/2$ in such an exceptional case

$$\begin{aligned} r_t^1 &= 2[(r^1 + r^2) + (r^1 - r^2) \ln(r^1 - r^2)]r_x^1, & r_y^1 &= (r^1 - r^2)r_x^1, \\ r_t^2 &= 2[(r^1 + r^2) - (r^1 - r^2) \ln(r^1 - r^2)]r_x^2, & r_y^2 &= -(r^1 - r^2)r_x^2, \end{aligned}$$

where

$$u = 2(r^1 + r^2), \quad v = 2 \ln(r^1 - r^2), \quad \beta = \pm\infty.$$

Let us consider the above chain of transformations for the initial Kupershmidt chain ($\beta = 2$). In such a case $\varepsilon_{(0)} = 0$. This case is trivial. The corresponding gas dynamics is decomposed in a couple of separate so-called the Euler-Monge-Riemann-Hopf equations

$$r_t^1 = r^1 r_x^1, \quad r_t^2 = r^2 r_x^2.$$

The first iteration yields $\beta_{(1)} = -2$, $\varepsilon_{(1)} = 1$. This is the first nontrivial, but still linearizable case. This is the two-component linearly degenerate system (see [22]). The second iteration yields $\beta_{(2)} = 2/3$, $\varepsilon_{(2)} = -1$. This is the two-component chromatography system (see [22]). The corresponding hydrodynamic type system is also linearizable (see [22]). All other iterations yield $\varepsilon_{(2K-1)} = -\varepsilon_{(2K)} = K$. Corresponding hydrodynamic type systems are linearizable too (see [22]).

Let us consider the Gibbons–Tsarev system (19) written in the form (see [21])

$$\partial_i q^k = \frac{q^k \partial_i B^0}{q^i - q^k}, \quad \partial_{ik} B^0 = \frac{(q^i + q^k) \partial_i B^0 \partial_k B^0}{(q^i - q^k)^2}, \quad i \neq k.$$

Following [21] let us restrict our attention on two component hydrodynamic reductions such that

$$q^1 + q^2 = 0.$$

Then this Gibbons–Tsarev system can be integrated

$$B^0 = R_1(r^1) + R_2(r^2), \quad q^1 = -q^2 = \frac{1}{2}[R_1(r^1) - R_2(r^2)], \quad (51)$$

where $R_1(r^1)$ and $R_2(r^2)$ are arbitrary functions. The corresponding linear systems describing conservation law densities h and commuting flows w^k in the Riemann invariants (see details in [35])

$$\frac{\partial^2 h}{\partial R_1 \partial R_2} = \frac{\varepsilon}{R_1 - R_2} \left(\frac{\partial h}{\partial R_1} - \frac{\partial h}{\partial R_2} \right), \quad \frac{\partial^2 W}{\partial R_1 \partial R_2} = -\frac{\varepsilon}{R_1 - R_2} \left(\frac{\partial W}{\partial R_1} - \frac{\partial W}{\partial R_2} \right),$$

where $w^1 = \partial W / \partial R_1$ and $w^2 = \partial W / \partial R_2$ arise in the gas dynamics (see the first example in this section and [24]). These equations are nothing but famous the Euler–Darboux–Poisson equations. A lot of new particular solutions also can be found in [32]; all possible transformations of the first order are described in [1].

Indeed, the substitution (51) in the hydrodynamic type system (18)

$$r_t^1 = \left(q^1 + \frac{B^0}{\beta}\right) r_x^1, \quad r_t^2 = \left(q^2 + \frac{B^0}{\beta}\right) r_x^2$$

leads to the ideal gas dynamics (50). Its first negative commuting flow (see (44))

$$r_{t^{-1}}^1 = \frac{(\mathbf{H}_{-1})^{\beta-1}}{q^1} r_x^1, \quad r_{t^{-1}}^2 = \frac{(\mathbf{H}_{-1})^{\beta-1}}{q^2} r_x^2$$

leads to the nonlinear elasticity equation (50), where $\mathbf{H}_{-1} = [R_1(r^1) + R_2(r^2)]^{2/\beta}$ is a consequence of

$$\partial_k \ln \mathbf{H}_{-1} = \frac{\partial_k B^0}{\beta q^k},$$

which is a result of the compatibility conditions $\partial_t(r_{t^{-1}}^1) = \partial_{t^{-1}}(r_t^1)$.

10 Special values of index β

The Kupershmidt hydrodynamic lattice determined by a special set of the parameters $\beta = 1/N$ (where N is an integer) was derived by M. Blaszak in [4]. If $\beta = L/M$ (where L and M are integers), then the function $F(w)$ in (46) can be integrated via elementary functions. Thus, generating functions of conservation laws and commuting flows (see the section 7) contain logarithmic parts. For instance,

$$\beta = -\frac{1}{2}, \quad \partial_{t^2} p = \partial_{t^0} [\ln p + \mathbf{H}_0 p^{1/2} + (\mathbf{H}_1 + \mathbf{H}_0^2) p],$$

$$\beta = -1, \quad \partial_{t^1} p = \partial_{t^0} (\ln p + \mathbf{H}_0 p), \quad \partial_\tau \mathbf{H}_{-2} = -\partial_{t^0} \left(\frac{\mathbf{H}_{-2}}{p} + \ln(\mathbf{H}_{-1}/p) \right),$$

$$\beta = 1, \quad \partial_{t^{-1}} p = \partial_{t^0} \ln(p/\mathbf{H}_{-1}), \quad \partial_\tau \mathbf{H}_0 = \partial_{t^0} \ln p,$$

$$\beta = 1/2, \quad \partial_{t^{-2}} p = \partial_{t^0} \left(\ln(p/\mathbf{H}_{-1}) - \frac{\mathbf{H}_{-2}}{\mathbf{H}_{-1}} (p/\mathbf{H}_{-1})^{1/2} \right), \quad \partial_\tau \mathbf{H}_1 = \partial_{t^0} [\ln p + \mathbf{H}_0 p^{-1/2}].$$

Let us consider *infinitely many component reductions* of the Benney hierarchy (see (20)). The simplest example is the dispersionless limit of BKP hierarchy (see, for instance, [5]). The first higher commuting flow of the Benney hierarchy is

$$\partial_{t^2} A^k = \frac{1}{3} [k A^{k+n-1} \partial_x + n \partial_x A^{k+n-1}] \frac{\partial \mathbf{H}_3}{\partial A^n},$$

where $\mathbf{H}_3 = A^3 + 3A^0 A^1$. Let us split this hydrodynamic chain

$$A_{t^2}^k = A_x^{k+2} + A^0 A_x^k + k A^{k-1} A_x^1 + (k+1) A^k A_x^0, \quad k = 0, 1, \dots$$

on the *even* and *odd* sub-chains

$$A_{t^2}^{2k} = A_x^{2k+2} + A^0 A_x^{2k} + 2k A^{2k-1} A_x^1 + (2k+1) A^{2k} A_x^0, \quad k = 0, 1, \dots$$

$$A_{t^2}^{2k+1} = A_x^{2k+3} + A^0 A_x^{2k+1} + (2k+1) A^{2k} A_x^1 + 2(k+1) A^{2k+1} A_x^0, \quad k = 0, 1, \dots$$

It is easy to see, that the infinitely many component reduction $A^{2k+1} = 0$, $k = 0, 1, \dots$ is compatible with the above chain

$$\partial_{t^2} \tilde{A}_{(2)}^k = \partial_x \tilde{A}_{(2)}^{k+1} + \tilde{A}_{(2)}^0 \partial_x \tilde{A}_{(2)}^k + (2k+1) \tilde{A}_{(2)}^k \partial_x \tilde{A}_{(2)}^0, \quad k = 0, 1, \dots,$$

where $\tilde{A}_{(2)}^k \equiv A^{2k}$. The Riemann mapping for the Benney hierarchy

$$\lambda = \mu + \frac{A^0}{\mu} + \frac{A^1}{\mu^2} + \frac{A^2}{\mu^3} + \dots \quad (52)$$

(under the above constraint $A^{2k+1} = 0$) reduces to

$$\lambda = \mu + \frac{\tilde{A}_{(2)}^0}{\mu} + \frac{\tilde{A}_{(2)}^1}{\mu^3} + \frac{\tilde{A}_{(2)}^2}{\mu^5} + \dots$$

The inverse series to (52)

$$\mu = \lambda - \frac{\mathbf{H}_1}{\lambda} - \frac{\mathbf{H}_2}{\lambda^2} - \frac{\mathbf{H}_3}{\lambda^3} - \dots$$

reduced to

$$\mu = \lambda - \frac{\tilde{\mathbf{H}}_1^{(2)}}{\lambda} - \frac{\tilde{\mathbf{H}}_2^{(2)}}{\lambda^3} - \frac{\tilde{\mathbf{H}}_3^{(2)}}{\lambda^5} - \dots$$

also means that $\tilde{A}_{(2)}^k$ and $\tilde{\mathbf{H}}_1^{(2)}$ no longer depend on *odd* “times” t^{2k+1} . Let us call *even* “times” $t^{2k} = y^k$ (i.e. $x = t^0 = y^0$). The corresponding Gibbons equation is

$$\lambda_{y^1} - (\mu^2 + \tilde{A}_{(2)}^0) \lambda_x = \frac{\partial \lambda}{\partial \mu} \left[\mu_{y^1} - \partial_x \left(\frac{\mu^3}{3} + \tilde{A}_{(2)}^0 \mu \right) \right].$$

The second example can be obtained in a similar way. The second higher commuting flow of the Benney hierarchy is

$$\partial_{t^3} A^k = \frac{1}{4} [k A^{k+n-1} \partial_x + n \partial_x A^{k+n-1}] \frac{\partial \mathbf{H}_4}{\partial A^n},$$

where $\mathbf{H}_4 = A^4 + 4A^0 A^2 + 2(A^1)^2 + 2(A^0)^3$. Let us split this hydrodynamic chain

$$A_{t^3}^k = A_x^{k+3} + 2A^0 A_x^{k+1} + A^1 A_x^k + k A^{k-1} A_x^2 + (k+1) A^k A_x^1 + [(k+2) A^{k+1} + 3k A^{k-1} A^0] A_x^0, \quad k = 0, 1, \dots$$

on the *three* sub-chains $A_{t^3}^{3k} = A_x^{3k+3} + \dots$, $A_{t^3}^{3k+1} = A_x^{3k+4} + \dots$, $A_{t^3}^{3k+2} = A_x^{3k+5} + \dots$. Then the infinitely many component reduction $A^{3k} = A^{3k+2} = 0$, $k = 0, 1, \dots$ is compatible with the above chain

$$\partial_{t^3} \tilde{A}_{(3)}^k = \partial_x \tilde{A}_{(3)}^{k+1} + \tilde{A}_{(3)}^0 \partial_x \tilde{A}_{(3)}^k + (3k+2) \tilde{A}_{(3)}^k \partial_x \tilde{A}_{(3)}^0, \quad k = 0, 1, \dots,$$

where $\tilde{A}_{(3)}^k \equiv A^{3k+1}$. The Riemann mapping (52) reduces to

$$\lambda = \mu + \frac{\tilde{A}_{(3)}^0}{\mu^2} + \frac{\tilde{A}_{(3)}^1}{\mu^5} + \frac{\tilde{A}_{(3)}^2}{\mu^8} + \dots$$

The reduced inverse series

$$\mu = \lambda - \frac{\tilde{\mathbf{H}}_0^{(2)}}{\lambda^2} - \frac{\tilde{\mathbf{H}}_1^{(2)}}{\lambda^5} - \frac{\tilde{\mathbf{H}}_2^{(2)}}{\lambda^8} - \dots$$

also means that $\tilde{A}_{(3)}^k$ and $\tilde{\mathbf{H}}_0^{(3)}$ no longer depend on “times” t^{3k+1} and t^{3k+2} . Let us call “times” $t^{3k} = z^k$ (i.e. $x = t^0 = z^0$). The corresponding Gibbons equation is

$$\lambda_{z^1} - (\mu^3 + \tilde{A}_{(3)}^0)\lambda_x = \frac{\partial \lambda}{\partial \mu} \left[\mu_{z^1} - \partial_x \left(\frac{\mu^4}{4} + \tilde{A}_{(3)}^0 \mu \right) \right].$$

Thus, it is easy to generalize the above examples. Let us introduce $\tilde{A}_{(N)}^k \equiv A^{N(k+1)-2}$, where N is an arbitrary natural number. All other moments vanish. Such infinitely many component reduction is connected with the Riemann mapping

$$\lambda = \mu + \frac{\tilde{A}_{(N)}^0}{\mu^{N-1}} + \frac{\tilde{A}_{(N)}^1}{\mu^{2N-1}} + \frac{\tilde{A}_{(N)}^2}{\mu^{3N-1}} + \dots$$

The corresponding hydrodynamic chain

$$\partial_{t^N} \tilde{A}_{(N)}^k = \partial_x \tilde{A}_{(N)}^{k+1} + \tilde{A}_{(N)}^0 \partial_x \tilde{A}_{(N)}^k + [(k+1)N - 1] \tilde{A}_{(N)}^k \partial_x \tilde{A}_{(N)}^0, \quad k = 0, 1, 2, \dots$$

satisfies the Gibbons equation

$$\lambda_{\tau^1} - (\mu^N + \tilde{A}_{(N)}^0)\lambda_x = \frac{\partial \lambda}{\partial \mu} \left[\mu_{\tau^1} - \partial_x \left(\frac{\mu^{N+1}}{N+1} + \tilde{A}_{(N)}^0 \mu \right) \right].$$

All moments $\tilde{A}_{(N)}^k$ are functions of ‘times’ $\tau^k \equiv t^{kN}$, $k = 0, 1, 2, \dots$. The substitution $B_{(1-1/N)}^k \equiv N \tilde{A}_{(N)}^k$ yields the Kupershmidt hydrodynamic chain (12). Thus, infinitely many discrete values $\beta = N$ can be obtained directly from the Benney hierarchy (20) by the above described degeneration.

11 Quasilinear equations and their particular solutions

The extended Kupershmidt lattice is connected with an infinite set of 2+1 quasilinear equations, which can be obtained by an appropriate elimination of the conservation law densities \mathbf{H}_k and auxiliary time variables t^n . Without loss of generality let us restrict

our consideration for simplicity on two first 2+1 quasilinear equations (see generating functions of conservation laws and commuting flows in the section 7)

$$\begin{aligned}\partial_\tau \mathbf{H}_0 &= \partial_{t^0} \frac{p^{\beta-1}}{\beta-1}, & \partial_\tau \mathbf{H}_{-1} &= -\partial_{t^0} \frac{\mathbf{H}_{-1}}{p}, \\ \partial_{t^1} p &= \partial_{t^0} \left(\frac{p^{\beta+1}}{\beta+1} + \mathbf{H}_0 p \right), & \partial_{t^{-1}} p &= \partial_{t^0} \left(\frac{(p/\mathbf{H}_{-1})^{1-\beta}}{1-\beta} \right).\end{aligned}$$

Substituting the asymptotics (48) one can obtain infinitely many 2+1 quasilinear equations, where the first of them are

$$\Phi_{xy} = \Phi_{tt} + (\beta-1)\Phi_x\Phi_{xt} + (\Phi_t - \beta\Phi_x^2/2)\Phi_{xx},$$

$$\Phi_{\sigma t} = (\beta-1)\Phi_\sigma\Phi_{xx} + \Phi_x\Phi_{\sigma x},$$

where $\mathbf{H}_0 = \Phi_x$, $\mathbf{H}_1 = \Phi_t + (\beta-2)\Phi_x^2/2$, $(\mathbf{H}_{-1})^{\beta-1} = (\beta-1)\Phi_\sigma$ and $y = t^2$, $\sigma = t^{-1}$. These 2+1 quasilinear equations (for $\beta = 1/N$) were derived in [4]; the first of them is the so called “ N -dmKP”, the second equation is the so called “ N -dDym”. These 2+1 quasilinear equations also can be derived from the compatibility conditions $\partial_{t^1}(\partial_{t^2}p) = \partial_{t^2}(\partial_{t^1}p)$ and $\partial_{t^1}(\partial_{t^{-1}}p) = \partial_{t^{-1}}(\partial_{t^1}p)$. The third equation (for $\beta = 1/N$) is the so called “ N -dToda”)

$$(\beta-1)^{1/(\beta-1)}\Phi_{\sigma\sigma} = (\Phi_\sigma)^{-1/(\beta-1)}\Phi_{xz} - \Phi_z(\Phi_\sigma)^{-\beta/(\beta-1)}\Phi_{\sigma x},$$

where $\mathbf{H}_{-2} = (\beta-1)^{(2-\beta)/(\beta-1)}\Phi_z(\Phi_\sigma)^{(2-\beta)/(\beta-1)}$ and $z = t^{-2}$ was derived in [21] and can be obtained from the compatibility condition $\partial_{t^{-1}}(\partial_{t^{-2}}p) = \partial_{t^{-2}}(\partial_{t^{-1}}p)$. However, this third 2+1 quasilinear equation is equivalent the second 2+1 quasilinear equation (“ N -dDym”) up to the transformation $t^k \leftrightarrow \tilde{t}^{-1-k}$ described in the section 7.

The generalized hodograph method (see [35]) allows to construct a general solution for N component hydrodynamic reductions described by the Gibbons–Tsarev system (19). The Gibbons–Tsarev system has a general solution parameterized by N arbitrary functions of a single variable. We are not able to find this general solution at this moment. However, infinitely many particular solutions (parameterized by the hypergeometric function) are already known (see the section 4). Since the generating function of commuting flows (see (46)) is found for the Kupershmidt hydrodynamic chain and for all its hydrodynamic reductions, then particular solutions for the above 2+1 quasilinear equations can be found by the generalized hodograph method written in the Riemann coordinates r^k (see (18))

$$x + tv_{(1)}^i(\mathbf{r}) + yv_{(2)}^i(\mathbf{r}) + zv_{(3)}^i(\mathbf{r}) = w^i(\mathbf{r})$$

or via field variables (conservation law densities, see (17)) b^k

$$x\delta_k^i + tv_{(1)k}^i(\mathbf{b}) + yv_{(2)k}^i(\mathbf{b}) + zv_{(3)k}^i(\mathbf{b}) = w_k^i(\mathbf{b}), \quad (53)$$

where the corresponding hydrodynamic reductions

$$r_{t^n}^i = v_{(n)}^i(\mathbf{r})r_{t^0}^i \quad \Leftrightarrow \quad b_{t^n}^i = v_{(n)k}^i(\mathbf{b})b_{t^0}^k, \quad i, k = 1, 2, \dots, N, \quad n = \pm 1, \pm 2$$

are written below in the explicit form in the Riemann invariants

$$r_t^k = [(r^k)^\beta + \mathbf{h}_0]r_x^k, \quad r_y^k = [(r^k)^{2\beta} + (\beta + 1)\mathbf{h}_0(r^k)^\beta + \mathbf{h}_1 + (\mathbf{h}_0)^2]r_x^k,$$

$$r_\sigma^k = (\mathbf{h}_{-1})^{\beta-1}(r^k)^{-\beta}r_x^k, \quad r_z^k = [(\mathbf{h}_{-1})^{2\beta-1}(r^k)^{-2\beta} + (\beta - 1)\mathbf{h}_{-2}(\mathbf{h}_{-1})^{\beta-2}(r^k)^{1-\beta}]r_x^k,$$

and in the conservative form

$$b_t^i = \partial_x \left(\frac{(b^i)^{\beta+1}}{\beta + 1} + \mathbf{h}_0 b^i \right), \quad b_y^i = \partial_x \left(\frac{(b^i)^{2\beta+1}}{2\beta + 1} + \mathbf{h}_0 (b^i)^{\beta+1} + (\mathbf{h}_1 + (\mathbf{h}_0)^2) b^i \right),$$

$$b_\sigma^i = \partial_x \left(\frac{(b^i/\mathbf{h}_{-1})^{1-\beta}}{1 - \beta} \right), \quad b_z^i = \partial_x \left(\frac{(b^i/\mathbf{h}_{-1})^{1-2\beta}}{1 - 2\beta} - \mathbf{h}_{-2}(\mathbf{h}_{-1})^{\beta-2}(b^i)^{1-\beta} \right).$$

Without loss of generality let us restrict our consideration for simplicity on N component hydrodynamic reductions determined by the moment decomposition (2). Then N infinite series of the conservation law densities $h_n^{(k)}$ can be found by the Bürmann–Lagrange expansion (9) from the equation of the Riemann surface

$$\lambda = [1 - (\frac{b^i}{p^{(i)}})^\beta] \exp \left[\sum_{n=1}^{\infty} \frac{(-1)^n (\gamma - 1)!}{nn! (\gamma - 1 - n)!} [1 - (\frac{b^i}{p^{(i)}})^\beta]^n - \frac{1}{\varepsilon^i} \sum_{m \neq i} \varepsilon_m \sum_{n=0}^{\infty} \frac{(b^m/p^{(i)})^{\beta(n+\gamma)}}{n + \gamma} - \frac{\beta(p^{(i)})^{\beta(1-\gamma)}}{\varepsilon^i(1 - \gamma)} \right]$$

at the vicinity of the every puncture $p_0^{(i)} = b^i$:

$$h_1^{(i)} = \frac{b^i}{\beta} \exp \left[\frac{1}{\varepsilon_i} \left(\frac{\beta(b^i)^{\beta(1-\gamma)}}{1 - \gamma} + \sum_{m \neq i} \varepsilon_m \left(\frac{b^m}{b^i} \right)^{\beta\gamma} F \left(1, \gamma, \gamma + 1, \left(\frac{b^m}{b^i} \right)^\beta \right) \right) \right],$$

$$h_2^{(i)} = \frac{(h_1^{(i)})^2}{b^i} \left[\frac{1 - \beta}{2} + \beta\gamma + \frac{\beta(b^i)^{\beta(1-\gamma)}}{\varepsilon_i} \left(\beta - \sum_{m \neq i} \frac{\varepsilon_m (b^m)^{\beta\gamma}}{(b^i)^\beta - (b^m)^\beta} \right) \right], \dots$$

Since the generating function of commuting flows (replace $p \rightarrow b^i$ in (46)) is

$$b_{\tau(\zeta)}^i = -\frac{1}{\beta} \partial_x \left[\frac{b^i}{p(\zeta)} F \left(1, \frac{1}{\beta}, \frac{\beta + 1}{\beta}, \left(\frac{b^i}{p(\zeta)} \right)^\beta \right) \right],$$

then the Taylor expansion at the vicinity $p_0^{(k)} = b^k$ (see [28]) yields N infinite series of commuting flows

$$b_{t^{k,n}}^i = \partial_x w_{(k,n)}^i(\mathbf{b}), \quad i, k = 1, 2, \dots, N, \quad n = 0, 1, 2, \dots$$

For instance, the first such commuting flow is

$$b_{t^{i,0}}^k = -\frac{1}{\beta} \partial_x \left[\frac{b^k}{b^i} F \left(1, \frac{1}{\beta}, \frac{\beta + 1}{\beta}, \left(\frac{b^k}{b^i} \right)^\beta \right) \right], \quad k \neq i,$$

$$b_{t^{i,0}}^i = \frac{1}{\varepsilon_i} \partial_x \left[\frac{(b^i)^{1-\gamma}}{1 - \gamma} - \frac{1}{\beta} \sum_{n \neq i} \varepsilon_n \left(\frac{b^n}{b^i} \right)^{\beta\gamma} F \left(1, \gamma, \gamma + 1, \left(\frac{b^n}{b^i} \right)^\beta \right) \right].$$

Thus, the generalized hodograph method (53) yields infinitely many particular solutions

$$x\delta_k^i + t\partial_k \left(\frac{(b^i)^{\beta+1}}{\beta+1} + \mathbf{h}_0 b^i \right) + y\partial_k \left(\frac{(b^i)^{2\beta+1}}{2\beta+1} + \mathbf{h}_0 (b^i)^{\beta+1} + (\mathbf{h}_1 + (\mathbf{h}_0)^2) b^i \right) \\ + \sigma\partial_k \left(\frac{(b^i/\mathbf{h}_{-1})^{1-\beta}}{1-\beta} \right) + z\partial_k \left(\frac{(b^i/\mathbf{h}_{-1})^{1-2\beta}}{1-2\beta} - \mathbf{h}_{-2}(\mathbf{h}_{-1})^{\beta-2} (b^i)^{1-\beta} \right) = \partial_k w_{(m,n)}^i.$$

12 Conclusion and outlook

Solutions of 2+1 quasilinear equations and corresponding hydrodynamic chains depend on integrability of the Gibbons–Tsarev system (19). In this paper we present multi-parametric family of hydrodynamic reductions, whose Riemann surfaces are associated with the hypergeometric function ${}_2F_1(1, \gamma, \gamma+1, z)$. Taking into account another class of hydrodynamic reductions derived in [13] for the Benney hydrodynamic chain (20), whose Riemann surfaces are associated with hyperelliptic integrals, we believe that more general and complicated hydrodynamic reductions can be found.

However, the main result of this paper is the existence of an infinite set of local Hamiltonian structures. Since the sub-case $\beta = 1$ is connected with the Benney hydrodynamic chain (20) by the Miura type transformation, it means that the Benney hydrodynamic chain also has an infinite series of local Hamiltonian structures.

The Kupershmidt hydrodynamic lattice has four very interesting and important sub-cases: $\beta = 1$, $\beta = 2$, $\beta = \infty$ and $\beta = 0$. The first sub-case is the modified dKP hierarchy, the second sub-case is the dBKP hierarchy, the third case is a continuum limit of the 2DToda hierarchy, the fourth case we call the universal hierarchy (see [2] and [24]). Just the sub-case $\beta = 2$ can be easily extracted from this paper. In all other sub-cases all formulas have singularities for corresponding values of the index β . These cases $\beta = 1$, $\beta = \infty$ (and $\beta = 2$) determine the Egorov hydrodynamic chains connected by the aforementioned transformations and by the Miura type transformation. They are considered in details in [31].

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References

- [1] *A.V. Aksenov*, Symmetries and relations between solutions of a class of Euler–Poisson–Darboux equations. (Russian) Dokl. Akad. Nauk. (Reports of RAS), **381** No. 2 (2001) 176–179.

- [2] *L.M. Alonso, A.B. Shabat*, Energy-dependent potentials revisited: A universal hierarchy of hydrodynamic type, *Phys. Lett. A*, **299** No. 4 (2002) 359-365.
- [3] *D.J. Benney*, Some properties of long non-linear waves, *Stud. Appl. Math.*, **52** (1973) 45-50.
- [4] *M. Błaszak*, Classical R -matrices on Poisson algebras and related dispersionless systems *Phys. Lett. A*, **297** (2002), 191-195. *M. Błaszak, B.M. Szablikowski*, Classical R -matrix theory of dispersionless systems: I. (1+1)-dimension theory, *J. Phys. A: Math. Gen.*, **35** (2002) 10325-10344. *M. Błaszak, B.M. Szablikowski*, Classical R -matrix theory of dispersionless systems: II. (2+1)-dimension theory, *J. Phys. A: Math. Gen.*, **35** (2002) 10345-10364. *M. Błaszak, B.M. Szablikowski*, Meromorphic Lax representations of (1+1)-dimensional multi-Hamiltonian dispersionless systems, submitted for publication, arXiv: nlin.SI/0510068. *B.M. Szablikowski*, Gauge transformation and reciprocal link for (2+1)-dimensional integrable field systems, *J. Nonlinear Math. Phys.*, **13** (2006) 117-128.
- [5] *L.V. Bogdanov, B.G. Konopelchenko*, Symmetry constraints for dispersionless integrable equations and systems of hydrodynamic type, *Phys. Lett. A*, **330** (2004) 448-459.
- [6] *I.Ya. Dorfman*, Dirac structures and integrability of nonlinear evolution equations; Nonlinear Science: Theory and Applications, John Wiley & Sons, New York (1993) 176 pp.
- [7] *B.A. Dubrovin, S.P. Novikov*, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, *Soviet Math. Dokl.*, **27** (1983) 665-669. *B.A. Dubrovin, S.P. Novikov*, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, *Russian Math. Surveys*, **44** No. 6 (1989) 35-124.
- [8] *E.V. Ferapontov*, Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, *Amer. Math. Soc. Transl. (2)*, **170** (1995) 33-58.
- [9] *E.V. Ferapontov, D.G. Marshall*, Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, *arXiv:nlin.SI/0505013*.
- [10] *E.V. Ferapontov, O.I. Mokhov*, Nonlocal Hamiltonian operators of hydrodynamic type that are connected with metrics of constant curvature, *Russian Math. Surveys*, **45** No. 3 (1990) 218-219.
- [11] *J. Gibbons*, Collisionless Boltzmann equations and integrable moment equations, *Physica D*, **3** (1981) 503-511.
- [12] *J. Gibbons, S.P. Tsarev*, Reductions of Benney's equations, *Phys. Lett. A*, **211** (1996) 19-24. *J. Gibbons, S.P. Tsarev*, Conformal maps and reductions of the Benney equations, *Phys. Lett. A*, **258** (1999) 263-270.

- [13] *J. Gibbons, L.A. Yu*, The initial value problem for reductions of the Benney equations, *Inverse Problems*, **16** No. 3 (2000) 605-618, *L.A. Yu*, Waterbag reductions of the dispersionless discrete KP hierarchy, *J. Phys. A: Math. Gen.*, **33** (2000) 8127–8138.
- [14] *B.A. Kupershmidt*, Deformations of integrable systems, *Proc. Roy. Irish Acad. Sect. A*, **83** No. 1 (1983) 45-74. *B.A. Kupershmidt*, Normal and universal forms in integrable hydrodynamical systems, *Proceedings of the Berkeley-Ames conference on nonlinear problems in control and fluid dynamics* (Berkeley, Calif., 1983), in *Lie Groups: Hist., Frontiers and Appl. Ser. B: Systems Inform. Control, II*, Math Sci Press, Brookline, MA, (1984) 357-378.
- [15] *B.A. Kupershmidt, Yu.I. Manin*, Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations, *Func. Anal. Appl.*, **12** No. 1 (1978) 25–37. *D.R. Lebedev, Yu.I. Manin*, Conservation laws and representation of Benney’s long wave equations, *Phys. Lett. A*, **74** No. 3,4 (1979) 154-156.
- [16] *B.A. Kupershmidt*, Hydrodynamic chains of Pavlov’s class, accepted in *Phys. Lett. A*.
- [17] *B.A. Kupershmidt*, Equations of long waves with a free surface III. The multidimensional case, *J. Nonlinear Math. Phys.*, **12** No. 4 (2005) 539-549. *B.A. Kupershmidt*, Extensions of 1-dimensional polytropic gas dynamics, *J. Nonlinear Math. Phys.*, **13** No. 1 (2006) 145-157.
- [18] *M.A. Lavrentiev, B.V. Shabat*, *Metody teorii funktsii kompleksnogo peremennogo* (Russian) [Methods of the theory of functions of a complex variable] Third corrected edition Izdat. “Nauka”, Moscow (1965) 716 pp.
- [19] *A.Ya. Maltsev, S.P. Novikov*, On the local systems Hamiltonian in the weakly non-local Poisson brackets, *Physica D*, **156** (2001) 53-80.
- [20] *M. Manas, L.M. Alonso, E. Medina*, Reductions of the dispersionless KP hierarchy, *Theor. Math. Phys.*, **133** No. 3 (2002) 1712-1721, *B. Konopelchenko, L.M. Alonso, E. Medina*, Quasiconformal mappings and solutions of the dispersionless KP hierarchy, *Theor. Math. Phys.*, **133** No. 2 (2002) 1529-1538.
- [21] *M. Manas*, S –functions, reductions and hodograph solutions of the r th dispersionless modified KP and Dym hierarchies, *J. Phys. A: Math. Gen.*, **37** (2004) 11191–11221. *M. Manas*, On the r th dispersionless Toda hierarchy: factorization problem, additional symmetries and some solutions, *J. Phys. A: Math. Gen.*, **37** (2004) 9195-9224.
- [22] *Y. Nutku*, On a new class of completely integrable nonlinear wave equations. Multi-Hamiltonian structure II, *J. Math. Phys.*, **28** No. 11 (1987) 2579–2585.
- [23] *M.V. Pavlov*, Classification of the Egorov hydrodynamic chains, *Theor. Math. Phys.*, **138** No. 1 (2004) 55-71.

- [24] *M.V. Pavlov*, Integrable hydrodynamic chains, J. Math. Phys., **44** No. 9 (2003) 4134-4156.
- [25] *M.V. Pavlov*, Integrable systems and metrics of constant curvature, J. Nonlinear Math. Phys., No. 9 Supplement 1 (2002) 173-191.
- [26] *M. V. Pavlov, S.P. Tsarev*, Three-Hamiltonian structures of the Egorov hydrodynamic type systems, Funct. Anal. Appl., **37** No. 1 (2003) 32-45.
- [27] *M.V. Pavlov*, Hydrodynamic chains and the classification of their Poisson brackets.
- [28] *M.V. Pavlov*, Algebro-geometric approach in the theory of integrable hydrodynamic type systems.
- [29] *M.V. Pavlov*, The Hamiltonian approach in the classification and the integrability of hydrodynamic chains.
- [30] *M.V. Pavlov*, Classification of integrable hydrodynamic chains and generating functions of conservation laws.
- [31] *M.V. Pavlov*, Explicit solutions of the WDVV equation determined by the “flat” hydrodynamic reductions of the Egorov hydrodynamic chains.
- [32] *M.V. Pavlov, Z. Popowicz*, Non-polynomial conservation law densities generated by the symmetry operators in some hydrodynamical models. J. Phys. A.: Math. and Gen., **36** (2003) 1-10.
- [33] *C. Rogers, W. F. Shadwick*, Bäcklund Transformations and their Applications, Academic Press (1982) NewYork.
- [34] *B.L. Rozhdestvenski, N.N. Yanenko*, Systems of quasilinear equations and their applications to gas dynamics. Translated from the second Russian edition by J. R. Schulenberger. Translations of Mathematical Monographs, 55. American Mathematical Society, Providence, RI, 1983; Russian ed. Nauka, (1968) Moscow.
- [35] *S.P. Tsarev*, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl., **31** (1985) 488–491. *S.P. Tsarev*, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya, **37** No. 2 (1991) 397–419.
- [36] *V.E. Zakharov*, Benney’s equations and quasi-classical approximation in the inverse problem method, Funct. Anal. Appl., **14** No. 2 (1980) 89-98. *V.E. Zakharov*, On the Benney’s Equations, Physica 3D, (1981) 193-200.